TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 5, Pages 2019–2058
S 0002-9947(01)02711-8
Article electronically published on January 10, 2001

UNCONDITIONAL STRUCTURES OF WEAKLY NULL SEQUENCES

S. A. ARGYROS AND I. GASPARIS

Abstract. The following dichotomy is established for a normalized weakly null sequence in a Banach space: Either every subsequence admits a convex block subsequence equivalent to the unit vector basis of c_0, or there exists a subsequence which is boundedly convexly complete.

1. Introduction

The semi-normalized weakly null sequences (i.e., sequences (x_j) in a Banach space converging weakly to zero and such that $\inf_j \|x_j\| > 0$) being a fundamental concept in the theory of Banach spaces have been studied extensively and several results about their structure, and the structure of the spaces spanned by them have been proved. We mention Bessaga and Pelczynski's theorem [B-P], that any such sequence has a subsequence which is Schauder basic, and Maurey and Rosenthal's examples [M-R] of weakly null sequences without unconditional basic subsequences. Both results are fundamental with enormous impact in the development of the theory.

After the appearance of Maurey and Rosenthal's examples, a number of results establishing certain forms of "restricted" unconditionality for weakly null sequences were obtained. We recall Elton's near unconditionality [E], [O1], the Schreier unconditionality, stated in [M-R] and also proved later by Odell [O2], and the Argyros-Mercourakis - Tsarpalias convex unconditionality [A-M-T]. Also, the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ [A-A], and the repeated averages hierarchy [A-M-T], determined the structure of those convex combinations of a weakly null sequence that tend to zero in norm.

The second and the third sections of the present paper are devoted to a unified approach of these results. Some of them are stated in a more general setting and the proofs, given here, are simpler than the existing ones. The fourth section contains a new dichotomy for weakly null sequences. We shall next explain our results related to this dichotomy and then present the results of the first two sections. We are interested in the isomorphic structure of subsequences of a given sequence (x_j). Therefore, in the sequel, by a weakly null sequence (x_j) we shall mean a normalized

Received by the editors November 20, 1998 and, in revised form, February 14, 2000.
2000 *Mathematics Subject Classification.* Primary 46B03; Secondary 06A07, 03E10.
Key words and phrases. Ramsey theory, weakly null sequence, convex block basis.

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sequence which is additionally bimonotone. That is, $\|x_j\| = 1$ and

$$\sup_n \max\left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\|, \left\| \sum_{i=n+1}^{\infty} a_i x_i \right\| \right\} \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\|,$$

for all choices of scalars (a_i). We shall use standard Banach space facts and terminology. Throughout this paper, X will denote a real infinite dimensional Banach space and B_X its closed unit ball. X^* stands for the Banach space of real-valued linear functionals on X which are continuous with respect to the norm topology. c_0 denotes the Banach space of real sequences tending to zero, under the supremum norm. ℓ_1 is the Banach space of absolutely summable real sequences, under the norm given by the sum of the absolute values of the coordinates. We mention here that in the sequel, we shall often identify the elements of ℓ_1 with signed measures on $\mathbb{N}$. A semi-normalized basic sequence (y_j) in X is called C-equivalent to the unit vector basis of c_0, if there exists a positive constant C such that $\| \sum_{j=1}^{n} a_j y_j \| \leq C \max_{j \leq n} |a_j|$, for every $n \in \mathbb{N}$, and all choices of scalars $(a_j)_{j=1}^{n}$.

Given (x_n) a sequence in a Banach space, a sequence (y_n) is called a *block subsequence* of (x_n), if there exist consecutive sets $F_i \subset \mathbb{N}$ (that is, $\max F_i < \min F_{i+1}$, $i \in \mathbb{N}$), and a sequence (a_i) of scalars such that for every $i \in \mathbb{N}$, $y_i = \sum_{n \in F_i} a_n x_n$. We then denote by $\operatorname{supp} y_i$, the support of y_i, that is the set $\{n \in F_i : |a_n| > 0\}$. We shall also adopt the notation $y_1 < y_2 < \cdots$ to indicate that (y_n) is a block subsequence of (x_n); (y_n) is called an *absolutely convex (resp. sub-convex) block subsequence* if (a_i) satisfies $\sum_{n \in F_i} |a_n| = 1$ (resp. $\sum_{n \in F_i} |a_n| \leq 1$). If in addition the a_i's are non-negative, then (y_n) is a *convex (resp. sub-convex) block subsequence* of (x_n).

For an infinite subset M of $\mathbb{N}$, we let $[M]$ (resp. $[M]^{<\infty}$) denote the set of its infinite (resp. finite) subsets. $[M]^k$, $k \in \mathbb{N}$, is the subset of $[M]^{<\infty}$ whose members have k elements. Finally, ω_1 denotes the first uncountable ordinal. We start with some definitions and notations.

Definition 1.1.

1. A sequence $s = (x_j)$ in a Banach space X is said to be series bounded if
$$\sup_n \left\| \sum_{j=1}^{n} x_j \right\| \stackrel{\text{def}}{=} b(s) < \infty.$$
2. A semi-normalized Schauder basic sequence $s = (x_j)$ is said to be semi-boundedly complete, (sb.c.), if for every sequence of scalars (a_j) such that $(a_j x_j)$ is series bounded, we have that $\lim_j a_j = 0$.

The basic definition related to our result is the following.

Definition 1.2. A semi-normalized Schauder basic sequence $s = (x_j)$ is said to be boundedly convexly complete, (b.c.c.), provided the following property holds for every sequence of scalars (a_j) such that $(a_j x_j)$ is series bounded: Given (F_j), a sequence of consecutive finite subsets of $\mathbb{N}$ such that $\sup_n \sum_{j \in F_n} |a_j| < \infty$, then $\lim_n \| \sum_{j \in F_n} a_j x_j \| = 0$.

It follows easily that every b.c.c. sequence (x_j) is also sb.c. In fact, every semi-normalized absolutely convex block subsequence of (x_j) is semi-boundedly complete. We mention here that as a consequence of our main result, Theorem 1.4,

one obtains that every weakly null sequence having the property that every semi-normalized absolutely convex block subsequence is semi-boundedly complete, admits a b.c.c. subsequence.

Our next proposition gives a simple characterization of b.c.c. sequences.

Proposition 1.3. *Let $s = (x_j)$ be a Schauder basic sequence and let $\mathbf{a} = (a_j)$ be a sequence of scalars. For every $n \in \mathbb{N}$, we define*

$$\tau_n(\mathbf{a}, s) = \sup\Big\{ \Big\| \sum_{j\in F} a_j x_j \Big\| : n \leq \min F, \sum_{j\in F} |a_j| \leq 1 \Big\},$$

and $\tau(\mathbf{a}, s) = \lim_n \tau_n(\mathbf{a}, s)$. $s = (x_j)$ is b.c.c. if and only if, for every sequence of scalars $\mathbf{a} = (a_j)$ such that $(a_j x_j)$ is series bounded we have that $\tau(\mathbf{a}, s) = 0$.

We omit the easy proof and observe that for a sequence $s = (x_j)$ we have that $\tau(\mathbf{a}, s) > 0$ if and only if there exists a sequence (F_n) of consecutive finite subsets of $\mathbb{N}$ such that $\sum_{j\in F_n} |a_j| \leq 1$, for all $n \in \mathbb{N}$, and $\inf_n \| \sum_{j\in F_n} a_j x_j \| > 0$.

Assuming s is weakly null and that $\mathbf{a} = (a_j)$ satisfies the stronger condition that the series $\sum_{j=1}^{\infty} a_j x_j$ converges in norm, it does not seem clear that $\tau(\mathbf{a}, s) = 0$. However, if (x_j) is convexly unconditional [A-M-T], it is guaranteed that for any such $\mathbf{a}$ we have that $\tau(\mathbf{a}, s) = 0$. The main result of the fourth section is the following.

Theorem 1.4. *For every weakly null sequence (x_j) one of the following two alternatives holds exclusively:*

1. *There exists a boundedly convexly complete subsequence.*
2. *Every subsequence admits a convex block subsequence equivalent to the unit vector basis of c_0.*

We recall that as a consequence of Elton's nearly unconditional theorem, one obtains a similar dichotomy for weakly null sequences where the two alternatives go as follows:

1. There exists a semi-boundedly complete subsequence.
2. Every subsequence admits a further subsequence equivalent to the unit vector basis of c_0.

Thus, our result may be considered as the continuation of Elton's theorem in the direction of a deeper search in the span of the sequence (x_j), for the existence of a block subsequence equivalent to the unit vector basis of c_0. Also, the alternative to the latter case is the existence of a restricted form of unconditionality which is, in an asymptotic sense, the near unconditionality for convex block subsequences.

Let us observe that if we assume that the sequence (x_j) is unconditional basic, then Theorem 1.4 follows easily by well known arguments. However, even in this case, our proof derives more information about the structure of the sequence. This is a consequence of our next result.

Theorem 1.5. *Let $s = (x_n)$ be a weakly null sequence having no subsequence which is b.c.c. There exist $N \in [\mathbb{N}]$, $\xi < \omega_1$ and a constant $C > 0$ such that the subsequence $(x_n)_{n\in N}$ is an ℓ_1^ξ spreading model, and for every $Q \in [N]$, $(\xi_n^Q \cdot s)_{n\in\mathbb{N}}$ is C-equivalent to the unit vector basis of c_0.*

In the above, $(\xi_n^Q \cdot s)_{n\in\mathbb{N}}$ is the sequence of repeated averages of order ξ of the sequence $(x_n)_{n\in Q}$. This concept will be explained in the next section.

Therefore, every weakly null sequence with no b.c.c. subsequence, has a subsequence which behaves similarly to the basis of Schreier's space X_ξ, for some countable ordinal ξ. Recall that X_ξ is defined as the completion of $(c_{00}, \|\cdot\|_\xi)$ where

$$\|x\|_\xi = \sup_{F\in S_\xi} \sum_{n\in F} |x(n)|$$

for $x = (x(n)) \in c_{00}$, the space of ultimately vanishing sequences. We do not know if the subsequence $(x_n)_{n\in M}$ resulting from Theorem 1.5 is actually equivalent to a subsequence of the unit vector basis of X_ξ for the corresponding ordinal ξ.

The statement of Theorem 1.4 reminds us of Rosenthal's remarkable dichotomy [R], for non-trivial weak-Cauchy sequences. (A weak Cauchy sequence is called non-trivial if it is non-weakly convergent.) We recall the statement of this theorem.

Theorem 1.6. *Every non-trivial weak Cauchy sequence in a (real or complex) Banach space has either a strongly summing subsequence or a convex block basis equivalent to the summing basis.*

Following [R], a weak Cauchy basic sequence (x_n) is said to be *strongly summing* (s.s.) provided that the scalar series $\sum_n a_n$ converges whenever $\sup_n \|\sum_{i=1}^n a_i x_i\|$ is finite. We also recall that the basic sequence (x_n) is equivalent to the summing basis provided that for every choice of scalars (a_n), the series $\sum_{n=1}^\infty a_n x_n$ converges if and only if the series $\sum_n a_n$ converges.

There are similarities but also differences between Rosenthal's dichotomy and ours. Their relation is discussed in detail in the last section of this paper where we also give a slight improvement of Rosenthal's result, Corollary 5.3, and establish the corresponding statement to Elton's dichotomy for non-trivial weak Cauchy sequences, Corollary 5.5.

The proof of Theorem 1.5 which immediately implies Theorem 1.4, is given in section 4 following Lemma 4.7. Our argument is of combinatorial nature depending heavily on results obtained in sections 2 and 3. Roughly speaking, the nature of this theorem enforces a delicate combination of the near unconditionality with the convex unconditionality and the results related to summability methods. Next we shall give a rough sketch of the proof of Theorem 1.5, indicating how the results obtained in Sections 2 and 3 are used in the argument.

Sketch of the proof of Theorem 1.5. Start with a normalized weakly null sequence $s = (x_n)$ and suppose that no subsequence of s is b.c.c. We can assume that s has no subsequence equivalent to the unit vector basis of c_0, or else the assertion of Theorem 1.5 trivially holds. Furthermore, Elton's dichotomy, Corollary 3.3, allows us to assume that the sequence (x_n^*) of functionals biorthogonal to (x_n) is weakly null in $[(x_n)]^*$. This fact is crucial for establishing the key Lemma 4.6. We shall effectively construct a convex block subsequence of (x_n) equivalent to the unit vector basis of c_0.

The first step is an easy stabilization argument given in Lemma 4.1. We obtain $M_0 \in [\mathbb{N}]$ and $\delta_0 > 0$ with the property that for every $L \in [M_0]$ there exist a scalar sequence $(a_m)_{m\in L}$ such that $(a_m x_m)_{m\in L}$ is series bounded by 1, and consecutive subsets (F_i) of L so that for every $i \in \mathbb{N}$ $(a_m)_{m\in F_i}$ is sub-convex with $\|\sum_{m\in F_i} a_m x_m\| > \delta_0$.

The main difficulty now is that changing from a subset L of M_0 to another subset N, the sub-convex block subsequence resulting from Lemma 4.1 applied on L

changes arbitrarily to a sub-convex block subsequence supported by N. Therefore, there is no obvious way to directly apply the infinite Ramsey theorem (Theorem 2.8). In order to overcome this difficulty, we make strong use of results on the ordinal complexity of weakly null sequences [A-A], [A-M-T]. We apply Theorem 2.13, Corollary 3.6 and Theorem 1.11 in Lemma 4.5, to obtain $M \in [\mathbb{N}]$, $\delta > 0$ and $1 \leq \xi < \omega_1$ satisfying the following properties:

1. $(x_m)_{m\in M}$ is an ℓ_1^ξ spreading model (Definition 2.1), yet no subsequence of $(x_m)_{m\in M}$ is an $\ell_1^{\xi+1}$ spreading model.
2. For every $N \in [M]$ we can find $P \in [N]$ so that for every finite sequence $(F_i)_{i=1}^n$ of consecutive maximal S_ξ subsets of P there exists a scalar sequence $(a_m)_{m\in N}$ which is ξ-faithful for $(\delta, (F_i)_{i=1}^n)$.

The latter means that

(i) $(a_m x_m)_{m\in N}$ is series bounded by 1 and $|a_m| \leq \frac{\delta}{2}$, for every $m \in N$.

(ii) $(a_m)_{m\in F_i}$ is sub-convex and $\| \sum_{m\in F_i} a_m x_m \| > \delta$, for all $i \leq n$.

It is a crucial fact for the entire proof that the first ordinal α such that (x_n) admits no subsequence which is an ℓ_1^α spreading model, is a successor ordinal $\alpha = \xi + 1$. This is the content of condition 1 above. On the other hand, condition 2 indicates that the semi-normalized convex block subsequence of (x_n) we are seeking, is related to the ordinal ξ. Indeed, our goal is to show that for some $L \in [M]$ the sequence $(\xi_i^L \cdot s)$ of repeated averages of order ξ of s, is actually equivalent to the unit vector basis of c_0. The repeated averages hierarchy $\{(\xi_n^L) : L \in [\mathbb{N}]\}$, enjoys very strong stability properties (**P3** and **P4** after the definition of the hierarchy) and for this reason it can be handled through Ramsey's theorem and generalizations of Elton's result (Theorem 1.9).

We next apply the combinatorial result of [AN-O] (Lemma 2.4 in our paper) to deduce that the sequence $(\xi_i^L \cdot s)$ is uniformly semi-normalized for all $L \in [M]$. The heart of the proof lies in establishing the following:

Given $\lambda > 0$, $\epsilon > 0$ and $N \in [M]$, there exist P, Q and L in $[N]$, $L \subset Q \subset P$, so that the following property is fulfilled:

If there exist $y^* \in B_{X^*}$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^n y^*(\xi_i^L \cdot s) > \lambda$, then there exist $x^* \in B_{X^*}$, a scalar sequence $(a_m)_{m\in P}$ and consecutive maximal S_ξ subsets $(F_i)_{i=1}^n$ of Q satisfying:

1. $\sum_{i=1}^n x^*(\xi_i^L \cdot s) > \lambda$, while $\sum_{m\in P\setminus\bigcup_{i=1}^n \operatorname{supp}\xi_i^L} |x^*(x_m)| < \epsilon$ (Lemma 3.2).
2. $(a_m)_{m\in P}$ is ξ-faithful for $(\delta, (F_i)_{i=1}^n)$ (Lemma 4.5).
3. $\sum_{m\in P\setminus\bigcup_{i=1}^n F_i} |x^*(x_m)| < 2\epsilon$ (Lemma 4.6 combined with 1 above).
4. $F_i \setminus \{\min F_i\} \subset \{m \in \operatorname{supp}\xi_i^L : x^*(x_m) \geq \frac{1}{2}x^*(\xi_i^L \cdot s)\}$, $i \leq n$ (Lemma 4.7 combined with Proposition 2.15).

Conditions 1–4 yield that for $\lambda > \frac{5}{\delta}$ and $N \in [M]$, there exists $L \in [N]$ so that $\| \sum_{i=1}^n \xi_i^L \cdot s\| \leq \lambda$, for all $n \in \mathbb{N}$. We deduce from the infinite Ramsey theorem, that for some $L \in [M]$, $(\xi_i^L \cdot s)$ is equivalent to the unit vector basis of c_0.

We next proceed with the results of the first two sections of this paper. In section 2 we present a criterion for embedding the family $S_\xi^n(M)$, Definition 2.3, where $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$, into a hereditary family $\mathfrak{F}$ of finite subsets of $\mathbb{N}$. This criterion, Theorem 2.13, is related to the notion of (ξ, M, δ, n) large families, Definition 2.16, introduced in [A-M-T] and [A-F], and for the case $n = 1$ it roughly says that given a hereditary family $\mathfrak{F}$ of finite subsets of $\mathbb{N}$, there exists $M \in [\mathbb{N}]$ with $S_\xi(M)$ contained in $\mathfrak{F}$ provided the following property holds for some subset

$\mathcal{A}$ of the probability measures on $\mathbb{N}$: $\sup_{F\in\mathfrak{F}}\mu(F)>\delta>0$, for all $\mu\in\mathcal{A}$, and for every $L\in[\mathbb{N}]$, every $\epsilon>0$ and $\zeta<\xi$, there exists $\mu\in\mathcal{A}$ supported by L and such that $\sup_{F\in S_\zeta}\mu(F)<\epsilon$. We apply Theorem 2.13 in section 3 in order to obtain a simpler proof (bypassing the use of the strong Cantor-Bendixson index) of the following dichotomy established in [A-M-T].

Theorem 1.7. *For a weakly null sequence $s=(x_n)$ in a Banach space and $1\leq\xi<\omega_1$, one of the following holds exclusively:*

1. *(x_n) is ξ-convergent.*
2. *There exists $M\in[\mathbb{N}]$, $M=(m_i)_{i\in\mathbb{N}}$, such that (x_{m_i}) is an l_1^ξ spreading model.*

ξ-convergence is introduced in Definition 2.2. The concept of an l_1^ξ spreading model is given in Definition 2.1.

It is perhaps worth noting that the summability methods introduced in [A-M-T] form the natural ordinal analogs of the Cesaro summability. These methods have been already employed in modern Banach space theory, [A-F], [O-S] and it is possible that they can be applied to other branches of mathematics as well. The proof of Theorem 1.7 given here is accessible to non-specialists.

Theorem 1.7 yields

Corollary 1.8. *Let (x_n) be a weakly null sequence which is an l_1^ξ spreading model for some ordinal $\xi<\omega_1$, yet no subsequence of (x_n) is an $l_1^{\xi+1}$ spreading model. Then there exists a semi-normalized convex block subsequence of (x_n) which is Cesaro summable.*

Let us remark here that the hypothesis of Corollary 1.8 is satisfied for any weakly null sequence in a Banach space whose Szlenk index [Sz], is equal to ω^n, $n\in\mathbb{N}$. Thus we generalize the result of Alspach and Odell, [A-O], who established Corollary 1.8 for weakly null sequences in $C(\omega^{\omega^n})$, $n\in\mathbb{N}$.

In the third section we give a simpler proof and a generalization of Elton's nearly unconditional theorem, [E], [O1]. More precisely we show

Theorem 1.9. *Let $s=(x_n)$ be a weakly null sequence in a Banach space and $\xi<\omega_1$. There exists $M\in[\mathbb{N}]$ such that for every $\delta\in(0,1]$ there exists a constant $C(\delta)>0$ so that the following property is satisfied: If $L\in[M]$, $n\in\mathbb{N}$ and $(a_i)_{i=1}^n$ are scalars in $[-1,1]$, then*

$$\Big\|\sum_{i\in F}a_i(\xi_i^L\cdot s)\Big\|\leq C(\delta)\max\Big\{\delta,\ \Big\|\sum_{i=1}^n a_i(\xi_i^L\cdot s)\Big\|\Big\},$$

for all $F\subset\{i\leq n:\ |a_i|\geq\delta\}$.

The proof of Theorem 1.9 is based on the combinatorial Lemma 3.2. The dual version of this lemma, Lemma 3.4, readily implies that every normalized weakly null sequence admits a convexly unconditional subsequence. This is one of the main results in [A-M-T]. It can be equivalently formulated as follows:

Theorem 1.10. *Let (x_n) be a weakly null sequence in a Banach space. There exists $M\in[\mathbb{N}]$, $M=(m_i)$, such that for all $\delta>0$, there exists a constant $C(\delta)>0$ so that the following property is satisfied: If $F\in[\mathbb{N}]^{<\infty}$ and $(\lambda_i)_{i\in F}$ are scalars with $\|\sum_{i\in F}\lambda_i x_{m_i}\|>\delta$ and $\sum_{i\in F}|\lambda_i|\leq 1$, then $\|\sum_{i=1}^\infty a_i x_{m_i}\|>C(\delta)$, for all choices of scalars $(a_i)_{i=1}^\infty\subset c_{00}$, with $\max_i|a_i|\leq 1$ and such that $|a_i|=|\lambda_i|$, for all $i\in F$.*

Another application of Lemma 3.4 is on the S_ξ unconditionality of l_1^ξ spreading models. Recall that the sequence (x_n) is said to be S_ξ unconditional, if there exists a constant $C > 0$ such that

$$\left\|\sum_{i\in F} a_i x_i\right\| \leq C\left\|\sum_{i=1}^{\infty} a_i x_i\right\|,$$

for every $F \in S_\xi$ and all choices of scalars $(a_i)_{i=1}^\infty \subset c_{00}$.

Theorem 1.11. *Let (x_n) be a weakly null sequence and $\xi < \omega_1$. Assume that (x_n) is an l_1^ξ spreading model. There exists $M \in [\mathbb{N}]$, $M = (m_i)$, such that (x_{m_i}) is S_ξ unconditional.*

We also obtain Corollary 3.7, the result on Schreier unconditionality [M-R], [O2], that every normalized weakly null sequence admits, for every $\epsilon > 0$, a subsequence which is $2 + \epsilon$ S_1 unconditional.

Our final results, Theorem 1.12 and Corollary 1.13, concern the duality between ${c_0}^\xi$ (Definition 3.8) and l_1^ξ spreading models, and the concept of the hereditary ξ Dunford-Pettis property (Definition 3.9).

Notation. If $M \in [\mathbb{N}]$ and (x_n) is a sequence in X, then we let X_M denote the closed linear span of the subsequence $(x_n)_{n\in M}$.

The duality between ${c_0}^\xi$ and ℓ_1^ξ spreading models is described next.

Theorem 1.12. *Let (x_n) and (x_n^*) be normalized weakly null sequences in X and X^* respectively. Assume that for some $\epsilon > 0$ we have that $\inf_n |x_n^*(x_n)| \geq \epsilon$. Let $1 \leq \xi < \omega_1$. The following are equivalent:*

1. *For every $M \in [\mathbb{N}]$, there exists $L \in [M]$ such that $(x_n^*|X_M)_{n\in L}$ is an ℓ_1^ξ spreading model in X_M^*.*
2. *For every $M \in [\mathbb{N}]$, there exists $L \in [M]$ such that $(x_n)_{n\in L}$ is a ${c_0}^\xi$ spreading model.*

Corollary 1.13. *For a Banach space X and $1 \leq \xi < \omega_1$, the following are equivalent:*

1. *Every normalized weakly null sequence in X admits a subsequence which is a c_0^ξ spreading model.*
2. *X is hereditarily ξ-DP.*

Before closing this section we would like to mention that according to an unpublished result of Johnson, [O1], if every subsequence of a normalized weakly null sequence (x_n) admits a further subsequence which is series bounded, then there exists a subsequence of (x_n) equivalent to the unit vector basis of c_0. Theorem 1.4 immediately yields a generalization of Johnson's result as the following corollary shows.

Corollary 1.14. *Let (x_n) be a weakly null sequence in a Banach space. Assume that every subsequence of (x_n) admits a semi-normalized convex block subsequence which is series bounded. Then there exists a convex block subsequence of (x_n) equivalent to the unit vector basis of c_0.*

We must point out here that even under the stronger assumption of Corollary 1.14, the proof has to go through the arguments of the general case.

We wish to thank H. Rosenthal for useful discussions regarding this paper.

2. Large families

In this section we present a criterion, Theorem 2.13, for embedding the family $S_\xi^n(M)$, where $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$, into a hereditary family $\mathfrak{F}$ of finite subsets of $\mathbb{N}$. This criterion will be applied in section 3 in the proof of Theorem 1.7.

We shall now introduce some notation and terminology that will be frequently used in the sequel and state all necessary definitions of the concepts discussed in this paper. We first recall the definition of the generalized Schreier families introduced in [A-A]. It is convenient here to associate to each countable ordinal ξ, a sequence of successor ordinals $(\xi_n + 1)$ such that $\xi_n + 1 = \xi$, for all $n \in \mathbb{N}$, if ξ is a successor ordinal, while $(\xi_n + 1)$ strictly increases to ξ, if ξ is a limit ordinal. In the sequel we shall refer to $(\xi_n + 1)$ as the sequence of ordinals associated to ξ.

Notation. If F_1, F_2 are non-empty subsets of $\mathbb{N}$ with F_1 finite, we denote by $F_1 < F_2$ the relation $\max F_1 < \min F_2$. If μ, ν are finitely supported signed measures on $\mathbb{N}$, we also write $\mu < \nu$ if $\operatorname{supp}\mu < \operatorname{supp}\nu$.

The Schreier families. $\{S_\xi\}_{\xi<\omega_1}$ are defined by transfinite induction as follows:

$$S_0 = \big\{\, \{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}.$$

Suppose that the families S_α have been defined for all $\alpha < \xi$.

If $\xi = \zeta + 1$, we set

$$S_\xi = \left\{ F \in [\mathbb{N}]^{<\infty} : F = \bigcup_{i=1}^{n} F_i, F_i \in S_\zeta, i \le n, n \le F_1 < \cdots < F_n \right\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal, let $(\xi_n + 1)$ be the sequence of ordinals associated to ξ and set

$$S_\xi = \{F \in [\mathbb{N}]^{<\infty} : n \le \min F, \text{ and } F \in S_{\xi_n+1}, \text{ for some } n \in \mathbb{N}\}.$$

Definition 2.1. A sequence (x_n) in a Banach space is an l_1^ξ spreading model, $\xi < \omega_1$, if there exists a constant $C > 0$ such that

$$\Big\| \sum_{i\in F} a_i x_i \Big\| \ge C \sum_{i\in F} |a_i|$$

for every $F \in S_\xi$ and all choices of scalars $(a_i)_{i\in F}$.

The repeated averages hierarchy. This hierarchy was introduced in [A-M-T]. We let (e_n) denote the unit vector basis of ℓ_1. For every countable ordinal ξ and every $M \in [\mathbb{N}]$, we define a convex block subsequence $(\xi_n^M)_{n=1}^\infty$ of (e_n) by transfinite induction on ξ in the following manner: If $\xi = 0$, then $\xi_n^M = e_{m_n}$, for all $n \in \mathbb{N}$, where $M = (m_n)$. Assume that $(\zeta_n^M)_{n=1}^\infty$ has been defined for all $\zeta < \xi$ and $M \in [\mathbb{N}]$. Let $\xi = \zeta + 1$. Set

$$\xi_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \zeta_i^M,$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \cdots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max \operatorname{supp} \xi_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = \frac{1}{k_n}\sum_{i=1}^{k_n} \zeta_i^{M_n}.$$

If ξ is a limit ordinal, let $(\xi_n + 1)$ be the sequence of ordinals associated to ξ, and let $M \in [\mathbb{N}]$. Define

$$\xi_1^M = [\xi_{m_1} + 1]_1^M$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \cdots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max \operatorname{supp} \xi_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = [\xi_{k_n} + 1]_1^{M_n}.$$

The inductive definition of $(\xi_n^M)_{n=1}^\infty$, $M \in [\mathbb{N}]$ is now complete. The following properties are established in [A-M-T].

P1: $(\xi_n^M)_{n=1}^\infty$ is a convex block subsequence of (e_n) and $M = \bigcup_{n=1}^\infty \operatorname{supp} \xi_n^M$ for all $M \in [\mathbb{N}]$ and $\xi < \omega_1$.

P2: $\operatorname{supp} \xi_n^M \in S_\xi$, for all $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$.

P3: If $M, N \in [\mathbb{N}]$, $\xi < \omega_1$, and $\operatorname{supp} \xi_i^M = \operatorname{supp} \xi_i^N$, for $i \le k$, then $\xi_i^M = \xi_i^N$ for $i \le k$.

P4: If $\xi < \omega_1$, $\{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$, and $\{L_k : k \in \mathbb{N}\} \subset [\mathbb{N}]$, are such that $\operatorname{supp} \xi_{n_i}^{L_i} < \operatorname{supp} \xi_{n_{i+1}}^{L_{i+1}}$, for all $i \in \mathbb{N}$, then letting $L = \bigcup_{i=1}^\infty \operatorname{supp} \xi_{n_i}^{L_i}$, we have that $\xi_i^L = \xi_{n_i}^{L_i}$, for all $i \in \mathbb{N}$.

Properties **P3** and **P4** are called stability properties of the hierarchy $\{(\xi_n^M)_{n=1}^\infty : M \in [\mathbb{N}]\}$.

Next we recall the definition of (M, ξ) convergent sequences.

Notation. For a sequence $s = (x_n)$ in a Banach space and a vector $\mu = \sum_{n=1}^\infty a_n e_n$ of ℓ_1, we set $\mu \cdot s = \sum_{n=1}^\infty a_n x_n$.

Definition 2.2. A sequence $s = (x_n)$ in a Banach space is called (M, ξ) convergent, $\xi < \omega_1$, $M \in [\mathbb{N}]$, if $\lim_n \|\xi_n^M \cdot s\| = 0$. The sequence s is called ξ-convergent, if for every $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that s is (L, ξ) convergent, for all $L \in [N]$.

Remark. A sequence is $(M, \xi + 1)$ convergent if and only if it is (M, ξ) summable. The latter concept is defined in [A-M-T]. Therefore, the $\xi + 1$-convergence of a sequence is equivalent to its ξ-summability introduced in [A-M-T].

We continue this preliminary discussion with the notion of a hereditary family.

Definition 2.3.
1. A family $\mathfrak{F}$ of finite subsets of $\mathbb{N}$ is called hereditary, if for every $G \in \mathfrak{F}$ and $F \subset G$ we have that $F \in \mathfrak{F}$.
2. If $\mathfrak{F}$ is hereditary and $M \in [\mathbb{N}]$, $M = (m_i)$, then we define $\mathfrak{F}[M] = \{F \in \mathfrak{F} : F \subset M\}$, and $\mathfrak{F}(M) = \{M(F) : F \in \mathfrak{F}\}$, where $M(F) = \{m_i : i \in F\}$.
3. If $\mathfrak{F}$ is hereditary and $n \in \mathbb{N}$, then we set

$$\mathfrak{F}^n = \left\{\bigcup_{i=1}^n F_i : F_1 < \cdots < F_n \text{ and } F_i \in \mathfrak{F}, i \le n\right\}.$$

Remark. Evidently, if $\mathfrak{F}$ is hereditary, then so are the families $\mathfrak{F}[M]$, $\mathfrak{F}(M)$ and $\mathfrak{F}^n$ for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$. It is also easily seen, by induction, that the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ are hereditary.

We observe that for all $\xi < \omega_1$ and $M \in [\mathbb{N}]$, $S_\xi(M) \subset S_\xi[M]$. Note that the inverse inclusion does not hold. However, the following result of Androulakis and Odell [AN-O], shows that the two families are in a certain sense comparable.

Lemma 2.4. *For every $M \in [\mathbb{N}]$ there exists $N \in [M]$ such that for all $\xi < \omega_1$, $\{F \setminus \{\min F\} : F \in S_\xi[N]\} \subset S_\xi(M)$.*

The next three lemmas describe properties of the maximal members of S_ξ, $\xi < \omega_1$. Their proofs are easily obtained by transfinite induction and therefore we omit them.

Lemma 2.5. *Let $\xi < \omega_1$ and $F \in S_\xi$. The following are equivalent:*

1. *If $F \subset G$ and $G \in S_\xi$, then $F = G$. Thus, F is a maximal S_ξ set.*
2. *$F \cup \{n\} \notin S_\xi$, for all $n \in \mathbb{N}$ with $\max F < n$.*
3. *$F \cup \{n\} \notin S_\xi$, for some $n \in \mathbb{N}$ with $\max F < n$.*

Lemma 2.6. *Suppose that $F_1 < \cdots < F_n$ belong to $S_\xi[M]$, $M \in [\mathbb{N}]$. There exist $k \le n$ and $G_1 < \cdots < G_k$, maximal members of $S_\xi[M]$ with $\min G_1 = \min F_1$ and such that*

$$\bigcup_{i=1}^{k-1} G_i \subset \bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{k} G_i, \quad (G_0 = \emptyset).$$

Lemma 2.7. *Let $M \in [\mathbb{N}]$, $\xi < \omega_1$. There exists a (necessarily) unique sequence $\{F_n^\xi(M)\}_{n\in\mathbb{N}}$ of consecutive maximal S_ξ sets such that*

$$M = \bigcup_{n=1}^{\infty} F_n^\xi(M).$$

Remark.
1. It is easily seen that if $N = \bigcup_{n=1}^\infty F_{k_n}^\xi(M)$, where $k_1 < k_2 < \cdots$, then $F_n^\xi(N) = F_{k_n}^\xi(M)$, for all $n \in \mathbb{N}$.
2. Let $M = (m_i)$ and $N = (n_i)$ be infinite subsets of $\mathbb{N}$. Assume that for some $p \in \mathbb{N}$, we have that $m_i = n_i$, for all $i \le p$. If $F_k^\xi(M)$ is contained in $\{m_i : i \le p\}$, then, $F_i^\xi(M) = F_i^\xi(N)$, for all $i \le k$.

Notation. We let $\mathcal{M}$ denote the set of those signed measures on $\mathbb{N}$ whose variation does not exceed 1. Clearly, $\mathcal{M}$ is naturally identified with the ball of ℓ_1. If $\mu \in \mathcal{M}$ and $\xi < \omega_1$, we set

$$\|\mu\|_\xi = \sup\{|\mu|(F) : F \in S_\xi\}.$$

We would like to mention here that the proofs of the results of this paper rely essentially on an important principle of infinite combinatorics known as the infinite Ramsey theorem. This theorem was proved in several steps by Nash-Williams [NW], Galvin and Prikry [G-P] and Silver [Si]. Proofs of the infinite Ramsey theorem which are not model-theoretic, were given by Ellentuck [Ell], and recently by Tsarpalias [T]. We recall the statement of the theorem. $[\mathbb{N}]$ is endowed with the topology of the pointwise convergence.

Theorem 2.8. *Let A be an analytic subset of $[\mathbb{N}]$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that either $[L] \subset A$, or, $[L] \subset [M] \setminus A$.*

In the sequel, any set satisfying the conclusion of Theorem 2.8, will be called (completely) Ramsey.

Our next lemma is crucial for the proof of Theorem 2.13.

Lemma 2.9. *Let $\mathfrak{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, ξ a countable ordinal and $m \in \mathbb{N}$. For every $M \in [\mathbb{N}]$ there exists $N \in [M]$ such that either $\bigcup_{i=1}^m F_i^\xi(L) \in \mathfrak{F}$ for all $L \in [N]$, or, $\bigcup_{i=1}^m F_i^\xi(L) \notin \mathfrak{F}$ for all $L \in [N]$.*

Proof. Let

$$\Delta = \left\{ L \in [M] : \bigcup_{i=1}^m F_i^\xi(L) \in \mathfrak{F} \right\}.$$

It follows by the second remark after Lemma 2.7, that Δ is closed in $[M]$ and therefore Ramsey. Hence, there exists $N \in [M]$ such that either $[N] \subset \Delta$, or, $[N] \bigcap \Delta = \emptyset$. If the former, then $\bigcup_{i=1}^m F_i^\xi(L) \in \mathfrak{F}$, for all $L \in [N]$. If the latter, then $\bigcup_{i=1}^m F_i^\xi(L) \notin \mathfrak{F}$, for all $L \in [N]$. □

Definition 2.10. Suppose that $\mathcal{A} \subset \mathcal{M}$, $\mathfrak{F}$ is a hereditary family of finite subsets of $\mathbb{N}$, $\delta > 0$, $M \in [\mathbb{N}]$ and $\xi < \omega_1$.

1. We shall say that $\mathfrak{F}$ δ-norms $\mathcal{A}$, if $\sup_{F \in \mathfrak{F}} |\mu|(F) > \delta$, for all $\mu \in \mathcal{A}$.
2. $\mathcal{A}$ is called (ξ, M) large, if for every $L \in [M]$, $\zeta < \xi$ and $\epsilon > 0$ there exists $\mu \in \mathcal{A}$ such that $|\mu|(\mathbb{N} \setminus L) < \epsilon$ and $\|\mu\|_\zeta < \epsilon$.

We are now ready for the proof of the main result of this section. We first treat the case $n = 1$.

Theorem 2.11. *Let $\mathfrak{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$, $\xi < \omega_1$, and $\delta > 0$. Assume that there exists $\mathcal{A} \subset \mathcal{M}$ which is (ξ, M) large and δ-normed by $\mathfrak{F}$. There exists $N \in [M]$ such that $S_\xi(N) \subset \mathfrak{F}$.*

Proof. We first consider the case of ξ being a successor ordinal, say $\xi = \zeta + 1$. Let $P \in [M]$ and $m \in \mathbb{N}$. We claim that there exists $Q \in [P]$ such that

$$\bigcup_{i=1}^m F_i^\zeta(L) \in \mathfrak{F}, \text{ for all } L \in [Q].$$

Indeed, if this is not the case, we obtain through Lemma 2.9, $Q \in [P]$ such that

$$\bigcup_{i=1}^m F_i^\zeta(L) \notin \mathfrak{F}, \text{ for all } L \in [Q].$$

Since $\mathcal{A}$ is (ξ, M) large, there exists $\mu \in \mathcal{A}$ such that

$$\|\mu\|_\zeta < \frac{\epsilon}{m} \text{ and } |\mu|(\mathbb{N} \setminus Q) < \frac{\epsilon}{m},$$

where $0 < \epsilon < \frac{\delta}{2}$. We also have that $\mathfrak{F}$ δ-norms $\mathcal{A}$ and therefore, there exists $F_0 \in \mathfrak{F}$ such that $|\mu|(F_0) > \delta$. It follows that $|\mu|(F_0 \bigcap Q) > \delta - \epsilon$, and thus, since $\mathfrak{F}$ is hereditary, we can assume that $F_0 \subset Q$ and $|\mu|(F_0) > \delta - \epsilon$. Set

$$L = F_0 \cup \{q \in Q : q > \max F_0\}.$$

Clearly, $L \in [Q]$. We may choose $k \in \mathbb{N}$ minimal with respect to $F_0 \subset \bigcup_{i=1}^k F_i^\zeta(L)$. Now $k - 1 < m$, as $\bigcup_{i=1}^{k-1} F_i^\zeta(L) \subset F_0$ and so it belongs to $\mathfrak{F}$. Therefore, $k \leq m$ and

thus

$$\delta - \epsilon < |\mu|(F_0) \leq k\|\mu\|_\zeta < \epsilon, \text{ by the choice of } \mu,$$

contradicting the choice of ϵ. Hence our claim holds and we can inductively choose

$$P_1 \supset P_2 \supset \cdots, \text{ infinite subsets of } M \text{ such that}$$

$$\bigcup_{i=1}^{n} F_i^\zeta(L) \in \mathfrak{F}, \text{ for all } L \in [P_n] \text{ and } n \in \mathbb{N}.$$

Next choose $m_1 < m_2 < \cdots$ with $m_n \in P_n$, for all $n \in \mathbb{N}$. We set $N = (m_n)$ and claim that $S_\xi(N) \subset \mathfrak{F}$. Indeed, let $F \in S_\xi$ and assume that $\min F = n$. Then $F = \bigcup_{i=1}^k F_i$, where $F_1 < \cdots < F_k$ belong to S_ζ and $k \leq n$. Applying Lemma 2.6, we obtain a finite sequence $(G_i)_{i=1}^l$, $l \leq k$, of consecutive maximal S_ζ subsets of N with $\min G_1 = m_n$ and such that

$$\bigcup_{i=1}^{l-1} G_i \subset N(F) \subset \bigcup_{i=1}^{l} G_i.$$

Note that $G_i \in [P_n]$, for $i \leq l$ and so there exists $L \in [P_n]$ such that

$$F_i^\zeta(L) = G_i, \text{ for all } i \leq l.$$

We now obtain that $N(F) \in \mathfrak{F}$, as $l \leq n$. This completes the proof for the case of a successor ordinal ξ.

Now let ξ be a limit ordinal and assume that the assertion of the theorem holds for all ordinals smaller than ξ. Let $(\xi_n + 1)$ be the sequence of ordinals associated to ξ. We can now choose by the induction hypothesis,

$$N_1 \supset N_2 \supset \cdots \text{ infinite subsets of } M$$

such that $S_{\xi_n+1}(N_n) \subset \mathfrak{F}$, for all $n \in \mathbb{N}$. Suppose that $N_i = (m_k^i)_{k\in\mathbb{N}}$, for all $i \in \mathbb{N}$, and choose $m_1 < m_2 < \cdots$ such that $m_i \in N_i$ and $m_i > m_i^i$, for all $i \in \mathbb{N}$. Set $N = (m_i)$ and it is easy to see that $S_\xi(N) \subset \mathfrak{F}$. □

Notation. Let $s = (x_n)$ be a sequence in a Banach space X and $\epsilon > 0$. We set

$$\mathfrak{F}_\epsilon(s) = \{F \in [\mathbb{N}]^{<\infty} : \exists\, x^* \in B_{X^*} \text{ with } |x^*(x_n)| \geq \epsilon, \forall\, n \in F\}.$$

Clearly, $\mathfrak{F}_\epsilon(s)$ is a hereditary family.

Corollary 2.12. *Let $s = (x_n)$ be a sequence in a Banach space and $\delta > 0$. Let ξ be a countable ordinal and suppose that there exists a set $\mathcal{A}$ of absolutely sub-convex combinations of (x_n) such that:*

1. *$\|x\| > \delta$, for all $x \in \mathcal{A}$.*
2. *For every $L \in [\mathbb{N}]$, $\epsilon > 0$ and $\zeta < \xi$, there exists $x \in \mathcal{A}$, $x = \sum_{i=1}^\infty a_i x_i$, such that $\sum_{i\notin L} |a_i| < \epsilon$ and $\sum_{i\in F} |a_i| < \epsilon$, for all $F \in S_\zeta$.*

Then there exists $N \in [\mathbb{N}]$ such that $S_\xi(N) \subset \mathfrak{F}_{\frac{\delta}{2}}(s)$.

Theorem 2.13. *Let $\mathfrak{F}$ be a hereditary family, $\xi < \omega_1$, $M \in [\mathbb{N}]$, $\delta > 0$ and $n \in \mathbb{N}$. Assume that there exists $\mathcal{A} \subset \mathcal{M}$, consisting of finitely supported measures, which is (ξ, M) large. Assume further that if $\mu_1 < \cdots < \mu_n$ and $\mu_i \in \mathcal{A}$, for $i \leq n$, then there exists $G \in \mathfrak{F}$ such that $|\mu_i|(G) > \delta$, for all $i \leq n$. Then, there exists $N \in [M]$ such that $S_\xi^n(M) \subset \mathfrak{F}$.*

We shall need the following

Lemma 2.14. *Let $M \in [\mathbb{N}]$, $n \in \mathbb{N}$ and $\xi < \omega_1$. Suppose that*

$$S_\xi^n(M) = \bigcup_{i=1}^{k} \mathfrak{F}_i,$$

where $\mathfrak{F}_i$ is hereditary for all $i \leq k$. Then there exist $N \in [M]$ and $i_0 \leq k$ such that $S_\xi^n(N) \subset \mathfrak{F}_{i_0}$.

Proof. Define

$$\Delta_i = \left\{ L \in [\mathbb{N}] : M\left(\bigcup_{j=1}^{n} F_j^\xi(L) \right) \in \mathfrak{F}_i \right\}, \text{ for } i \leq k.$$

Evidently, Δ_i is closed in $[\mathbb{N}]$ and therefore Ramsey, for all $i \leq k$. Of course, by our assumption, we have that

$$[\mathbb{N}] = \bigcup_{i=1}^{k} \Delta_i.$$

We may now choose $P \in [\mathbb{N}]$ and $i_0 \leq k$ such that $[P] \subset \Delta_{i_0}$. Let $N = M(P)$, and clearly $S_\xi^n(N) \subset \mathfrak{F}_{i_0}$. □

Proof of Theorem 2.13. By induction on n. The case $n = 1$ has been settled in Theorem 2.11. So assume $n \geq 2$. Choose $\mathcal{B}$ a countable dense subset of $\mathcal{A}$. Clearly, $\mathcal{B}$ satisfies the same assumptions as $\mathcal{A}$ does in the hypothesis of Theorem 2.13. Therefore, without loss of generality, we shall assume that $\mathcal{A}$ itself is countable. Let (a_k) be an enumeration of the elements of $\mathcal{A}$. We claim that there exist:

1. A sequence (G_k) of elements of $\mathfrak{F}$ such that $G_k \subset \operatorname{supp} a_k$ and $|a_k|(G_k) > \delta$, for all $k \in \mathbb{N}$.
2. A decreasing sequence (M_k) of infinite subsets of M such that $\operatorname{supp} a_k < M_k$ and $G_k \cup F \in \mathfrak{F}$, for all $F \in S_\xi^{n-1}(M_k)$, and all $k \in \mathbb{N}$.

Indeed, suppose that $G_1, \cdots, G_{k-1}$ and $M_1 \supset \cdots \supset M_{k-1}$ satisfying 1 and 2 have been constructed. Let $G \in \mathfrak{F}$ such that $G \subset \operatorname{supp} a_k$ and $|a_k|(G) > \delta$. Define

$$\mathfrak{F}_G = \{F \in \mathfrak{F} : G < F \text{ and } G \cup F \in \mathfrak{F}\}.$$

Let $\Delta = \bigcup_G \mathfrak{F}_G$, where the union is taken over all possible subsets G of $\operatorname{supp} a_k$ which belong to $\mathfrak{F}$ and satisfy the relation $|a_k|(G) > \delta$. Of course, Δ is hereditary and the hypothesis of Theorem 2.13 is satisfied for the family Δ, the integer $n-1$ and the set of measures $\{a_i : a_k < a_i\}$. By the induction hypothesis there exists, for all $k \in \mathbb{N}$, $N_k \in [M_{k-1}]$ such that $\operatorname{supp} a_k < N_k$ and $S_\xi^{n-1}(N_k) \subset \Delta$. Next choose according to Lemma 2.14, $G_k \in \mathfrak{F}$ and $M_k \in [N_k]$ such that $G_k \subset \operatorname{supp} a_k$, $|a_k|(G_k) > \delta$ and $S_\xi^{n-1}(M_k) \subset \mathfrak{F}_{G_k}$. It follows now that $G_k \cup F \in \mathfrak{F}$, for all $F \in S_\xi^{n-1}(M_k)$. This completes the inductive construction and our claim holds.

Now let F be a finite subset of $\mathbb{N}$ and set

$$k_F = \min\{k \in \mathbb{N} : F \subset G_k\}.$$

We let $k_F = 0$, if F is not contained in G_k, for all $k \in \mathbb{N}$. Inductively we construct a sequence of positive integers, $m_1 < m_2 < \cdots$, in the following manner: Suppose that $m_1 < \cdots < m_d$ and $q_1 \leq \cdots \leq q_{d-1}$ have been constructed. (m_1 is chosen arbitrarily in M_1 and $q_0 = 0$.) We set

$$q_d = \max\big\{k_F : F \subset \{m_1, \cdots, m_d\}\big\} \vee q_{d-1}.$$

Next choose $m_{d+1} \in M_{q_d+1}$ such that $m_{d+1} > m_d$ and $m_{d+1} > m_{d+1}^{q_d+1}$, where $M_i = (m_j^i)_{j=1}^{\infty}$, for all $i \in \mathbb{N}$. Let $P = (m_i)$. We now claim that if $F \subset P$ is contained in G_k, for some $k \in \mathbb{N}$, then $F \cup H \in \mathfrak{F}$, for all $H \in S_\xi^{n-1}(P)$ such that $F < H$. Indeed, let $m_d = \max F$. Then $k_F \leq q_d$ and $F \subset G_{k_F}$. It suffices to show that $H \in S_\xi^{n-1}(M_{k_F})$ for all $H \in S_\xi^{n-1}(P)$ such that $F < H$. Our claim will then follow by condition 2 above. To this end, let $H \in S_\xi^{n-1}(P)$, such that $F < H$. Suppose that $H = \{m_{i_1}, \cdots, m_{i_t}\}$, where $\{i_1, < \cdots <, i_t\}$ belongs to S_ξ^{n-1}. Now, for all $r \leq t$,

$$m_{i_r} \geq m_{d+1} \text{ and thus } i_r \geq d+1.$$

So,

$$m_{i_r} > m_{i_r}^{q_{i_r-1}} \geq m_{i_r}^{q_d} \geq m_{i_r}^{k_F}, \text{ as } k_F \leq q_d.$$

Hence, $H \in S_\xi^{n-1}(M_{k_F})$, as claimed. Finally, consider the hereditary family

$$\mathfrak{D} = \bigcup_{k=1}^{\infty} \{F : F \subset G_k\}.$$

Clearly, the hypothesis of Theorem 2.11 is satisfied for $\mathfrak{D}$ and the set of measures $\{a_k|G_k : k \in \mathbb{N}\}$. We can thus find $N \in [P]$ such that $S_\xi(N) \subset \mathfrak{D}$. It is now easily verified that $S_\xi^n(N) \subset \mathfrak{F}$. □

Our next proposition will enable us to verify that the set of measures

$$\mathcal{A}_\xi = \{\xi_n^L : n \in \mathbb{N}, L \in [\mathbb{N}]\}$$

is $(\xi, \mathbb{N})$ large, for all $\xi < \omega_1$.

Proposition 2.15. *For every $M \in [\mathbb{N}]$, $\epsilon > 0$, and all ordinals $\zeta < \xi < \omega_1$, there exists $N \in [M]$ (depending on M, ϵ, ζ, ξ) such that $\|\xi_n^L\|_\zeta < \epsilon$, for all $L \in [N]$ and $n \in \mathbb{N}$.*

Proof. It suffices to show the following:

Claim. Let $\xi < \omega_1$. For all $\zeta < \xi$, $\epsilon > 0$ and $M \in [\mathbb{N}]$, there exists $L \in [M]$ such that $\|\xi_1^L\|_\zeta < \epsilon$.

Indeed, assuming our claim holds, we observe that the set $\{L \in [M] : \|\xi_1^L\|_\zeta < \epsilon\}$, is closed in $[M]$ and therefore Ramsey. Our claim now yields the existence of $N \in [M]$ such that $\|\xi_1^L\|_\zeta < \epsilon$, for every $L \in [N]$. By stability, we obtain the assertion of the proposition.

We shall prove our claim by transfinite induction on ξ. If $\xi = 1$, then $\zeta = 0$ and the claim is easily verified. Assuming our claim holds for all ordinals smaller than ξ, first let ξ be a limit ordinal. Let $(\xi_n + 1)$ be the sequence of ordinals associated to ξ. Suppose that $\zeta < \xi$ and choose $m \in M$ so that

$$\zeta < \xi_m \text{ and } \frac{1}{m} < \frac{\epsilon}{2}.$$

We apply the induction hypothesis on the ordinal ξ_m and the set $L_1 = \{i \in M : i \geq m\}$ to obtain $L_2, \cdots, L_m$, infinite subsets of L_1 such that

$$[\xi_m]_1^{L_1} < [\xi_m]_1^{L_2} < \cdots < [\xi_m]_1^{L_m} \quad \text{and} \quad \|[\xi_m]_1^{L_i}\|_\zeta < \frac{\epsilon}{2m}, \text{ for } 2 \leq i \leq m.$$

By stability property P_4, there exists $N \in [M]$ with $\min N = m$ and such that

$$[\xi_m]_i^N = [\xi_m]_1^{L_i}, \text{ for } i \leq m.$$

Now,

$$[\xi_m + 1]_1^N = \frac{1}{m}\sum_{i=1}^{m}[\xi_m]_i^N = \xi_1^N.$$

Hence,

$$\|\xi_1^N\|_\zeta \leq \frac{1}{m} + \frac{m-1}{2m}\epsilon < \epsilon.$$

So our claim holds if ξ is a limit ordinal.

Suppose now that $\xi = \alpha + 1$. If $\zeta < \alpha$, choose according to the induction hypothesis $L_1 \in [M]$ such that $\|\alpha_1^{L_1}\|_\zeta < \epsilon$. Let $m = \min \operatorname{supp} \alpha_1^{L_1}$ and choose again by the induction hypothesis, $L_2, \cdots, L_m$, infinite subsets of M such that

$$\alpha_1^{L_1} < \alpha_1^{L_2} < \cdots < \alpha_1^{L_m} \text{ and } \|\alpha_1^{L_i}\|_\zeta < \epsilon, \text{ for } 2 \leq i \leq m.$$

But once again, by stability, there exists $N \in [M]$ with $\min N = m$ and such that

$$\alpha_i^N = \alpha_1^{L_i}, \text{ for } i \leq m.$$

Now, $\xi_1^N = \frac{1}{m}\sum_{i=1}^m \alpha_i^N$, and thus, $\|\xi_1^N\|_\zeta < \epsilon$.

The final case to consider is when $\zeta = \alpha$. Let $(\beta_i + 1)$ be the sequence of ordinals associated to α. Choose $m \in M$ such that $\frac{1}{m} < \frac{\epsilon}{2}$. Set $L_1 = \{n \in M : n \geq m\}$. It follows that

$$[\beta_m + 1]_1^{L_1} = \alpha_1^{L_1}.$$

Let $k_1 = \max \operatorname{supp} \alpha_1^{L_1}$. Choose according to the induction hypothesis $L_2 \in [M]$ with $k_1 < \min L_2$ and such that

$$\|\alpha_1^{L_2}\|_{\beta_j} < \frac{\epsilon}{2k_1}, \text{ for all } j \leq k_1.$$

Let $k_2 = \max \operatorname{supp} \alpha_1^{L_2}$. Successive repetitions of the previous argument yield

$$\alpha_1^{L_1} < \alpha_1^{L_2} < \cdots < \alpha_1^{L_m} \text{ with } L_i \in [M] \text{ for } i \leq m,$$

such that if $k_i = \max \operatorname{supp} \alpha_1^{L_i}$ for $i \leq m$, then

$$\|\alpha_1^{L_i}\|_{\beta_j} < \frac{\epsilon}{2k_{i-1}}, \text{ for all } j \leq k_{i-1} \text{ and } 2 \leq i \leq m.$$

Stability now guarantees the existence of $L \in [M]$ with $\min L = m$ and such that

$$\alpha_i^L = \alpha_1^{L_i}, \text{ for all } i \leq m.$$

Now, $\xi_1^L = \frac{1}{m}\sum_{i=1}^m \alpha_i^L$ and it remains to show that $\|\xi_1^L\|_\alpha < \epsilon$. Indeed, let $G \in S_\alpha$ and choose $i_0 \leq m$ minimal with respect to $G \cap \operatorname{supp} \alpha_{i_0}^L \neq \emptyset$. Let $l = \min G$ and observe that $l \leq k_{i_0}$. Choose $p \leq l$ such that $G \in S_{\beta_p+1}$. There exist $q \leq l$ and $(G_j)_{j=1}^q$ consecutive members of S_{β_p} such that $G = \bigcup_{j=1}^q G_j$. Note that

$$q \leq l \leq k_{i_0} \text{ and also } p \leq l \leq k_{i_0}.$$

Thus,

$$\|\alpha_i^L\|_{\beta_p} < \frac{\epsilon}{2k_{i_0}}, \text{ for } i_0 < i \leq m.$$

Therefore,

$$\alpha_i^L(G) \leq q\|\alpha_i^L\|_{\beta_p} < k_{i_0}\frac{\epsilon}{2k_{i_0}} = \frac{\epsilon}{2}, \text{ for } i_0 < i \leq m.$$

Hence,

$$\xi_1^L(G) = \frac{1}{m}\left[\alpha_{i_0}^L(G) + \sum_{i=i_0+1}^{m} \alpha_i^L(G)\right] < \frac{1}{m} + \frac{\epsilon}{2} < \epsilon.$$

and so $\|\xi_1^L\|_\alpha < \epsilon$. The proof of the claim is now complete. □

We next recall the concept of the (n, ξ, M, δ) large families [A-M-T], [A-F].

Definition 2.16. Let $M \in [\mathbb{N}]$, $\xi < \omega_1$, $\delta > 0$ and $n \in \mathbb{N}$. The hereditary family $\mathfrak{F}$ is called (n, ξ, M, δ) large provided that for all $N \in [M]$ there exists $F \in \mathfrak{F}$ such that $\xi_i^N(F) > \delta$, for all $i \leq n$.

Corollary 2.17. *Let $\mathfrak{F}$ be (n, ξ, M, δ) large. There exists $N \in [M]$ such that $S_\xi^n(N) \subset \mathfrak{F}$.*

Proof. Proposition 2.15 and the fact that $\mathfrak{F}$ is (n, ξ, M, δ) large, immediately yield that $\mathfrak{F}$ and $\mathcal{A}_\xi$ satisfy the hypothesis of Theorem 2.13. The assertion of the corollary now follows. □

3. A generalization of Elton's theorem

In the first part of this section we give the proof of Theorem 1.9 which extends Elton's nearly unconditional theorem [E], [O1]. The second part of section 3 is devoted to the proofs of the results concerning the summability methods (Theorem 1.7), the convex unconditionality (Theorem 1.10) and the S_ξ unconditionality (Theorem 1.11). We also discuss the duality between c_0^ξ and l_1^ξ spreading models (Theorem 1.12) as well as the ξ Dunford-Pettis property (Corollary 1.13).

Let $s = (x_n)$ be a weakly null sequence in the Banach space X. Recall here that s is additionally assumed to be a normalized, bimonotone, Schauder basic sequence. For this fixed sequence s and the countable ordinal ξ, we have the following:

Definition 3.1. Let $M \in [\mathbb{N}]$, $\lambda > 0$ and $k \in \mathbb{N}$. The functional $x^* \in B_{X^*}$ is said to be *ξ-good* for (λ, M, k), if $x^*(\xi_i^M \cdot s) \geq 0$, for all $i \leq k$, and $\sum_{i=1}^k x^*(\xi_i^M \cdot s) > \lambda$.

The main tool for proving Theorem 1.9 is the following combinatorial result.

Lemma 3.2. *Let $\epsilon > 0$, $\lambda > 0$ and $N \in [\mathbb{N}]$. There exists $M \in [N]$ satisfying the following property: If $L \in [M]$, $k \in \mathbb{N}$ and there exists $x^* \in B_{X^*}$ which is ξ-good for (λ, L, k), then there exists $y^* \in B_{X^*}$ which is ξ-good for (λ, L, k) and such that*

$$\sum_{n \in M \setminus \bigcup_{i=1}^k \operatorname{supp} \xi_i^L} |y^*(x_n)| < \epsilon.$$

Proof. First choose $(\epsilon_i)_{i=0}^\infty$, a sequence of positive scalars such that $\sum_{i=0}^\infty \epsilon_i < \epsilon$. Let $T_0 = \emptyset$ and $T_n = \{1, \cdots, n\}$, if $n \in \mathbb{N}$. By an n-tuple of positive integers $(m_i)_{i \in T_n}$, we shall either mean the empty tuple, if $n = 0$, or, that $m_1 < \cdots < m_n$, if $n \geq 1$. Now let $n \in \mathbb{N} \cup \{0\}$ and $F \subset T_n$. The n-tuple $(m_i)_{i \in T_n}$ and the infinite subset L of $\mathbb{N}$, $(L = (l_i))$, are said to satisfy property $(F - E_n)$, provided that $m_n < l_1$, if $n \geq 1$, and the following statement holds:

If $k \in \mathbb{N}$, and there exists $x^* \in B_{X^*}$ which is ξ-good for $(\lambda, \{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}, k)$, then there exists $y^* \in B_{X^*}$ which is ξ-good for $(\lambda, \{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}, k)$ and such that

$$\sum_{j \in F} |y^*(x_{m_j})| + |y^*(x_{l_1})| < \sum_{i=0}^{n} \epsilon_i.$$

Let us also say that $(m_i)_{i \in T_n}$ and L satisfy property (E_n), if they satisfy property $(F - E_n)$, for every $F \subset T_n$.

We shall inductively construct an increasing sequence $(m_n)_{n=1}^{\infty}$ of elements of N and a decreasing sequence $(M_n)_{n=0}^{\infty}$ of infinite subsets of N with $m_n \in M_{n-1}$, if $n \geq 1$, so that for every $n \in \mathbb{N} \cup \{0\}$, if $L \in [M_n]$, then $(m_i)_{i \in T_n}$ and L satisfy property (E_n).

The first inductive step is similar to the general one and therefore we shall not discuss it. Assume that $(m_i)_{i \in T_n}$ and $M_0 \supset \cdots \supset M_n$, infinite subsets of N, with $m_i \in M_{i-1}$ for $1 \leq i \leq n$ have been constructed so that if $i \leq n$ and $L \in [M_i]$, then $(m_j)_{j \in T_i}$ and L satisfy property (E_i). Let $m_{n+1} = \min M_n$. Fix $F \subset T_{n+1}$ and define

$$\Delta_F = \{L \in [M_n], L = (l_i) : (m_i)_{i \in T_{n+1}} \text{ and } L \text{ satisfy } (F - E_{n+1})\}.$$

Clearly, Δ_F is closed in $[M_n]$ and therefore Ramsey. Suppose that for some $P \in [M_n]$, $P = (p_i)$, we had that $[P] \cap \Delta_F = \emptyset$. Let $q \in \mathbb{N}$ and set

$$L_j = \{p_j\} \cup \{p_i : i > q\}, \text{ for all } j \leq q.$$

Since $L_j \notin \Delta_F$, for all $j \leq q$, there exist integers $(k_j)_{j=1}^{q}$ and functionals $(x_j^*)_{j=1}^{q}$ in B_{X^*} so that letting $R = \{m_i : i \in T_{n+1} \setminus F\} \cup \{p_i : i > q\}$, we have that for all $j \leq q$, x_j^* is ξ -good for (λ, R, k_j); moreover, if $y^* \in B_{X^*}$ is ξ-good for (λ, R, k_j), then

$$\sum_{i \in F} |y^*(x_{m_i})| + |y^*(x_{p_j})| \geq \sum_{i=0}^{n+1} \epsilon_i.$$

Next choose $j_0 \leq q$ such that $k_{j_0} = \min\{k_j : j \leq q\}$. We observe that if y^* is ξ-good for (λ, R, k_{j_0}) and $y^*(x_i) = 0$, for all $i > \max \operatorname{supp} \xi_{k_{j_0}}^R$, then y^* is also ξ-good for (λ, R, k_j), for every $j \leq q$.

Now let $t = \max F$ and note that

$$R = \{m_i : i \in T_{t-1} \setminus F\} \cup \{m_i : t < i \leq n+1\} \cup \{p_i : i > q\}.$$

By the induction hypothesis, since $(m_i)_{i \in T_{t-1}}$ and $\{m_i : t \leq i \leq n+1\} \cup \{p_i : i > q\}$ satisfy property (E_{t-1}), there exists $z^* \in B_{X^*}$ ξ-good for (λ, R, k_{j_0}) and such that

$$\sum_{i \in F \setminus \{t\}} |z^*(x_{m_i})| + |z^*(x_{m_t})| < \sum_{i=0}^{t-1} \epsilon_i.$$

Thus,

$$\sum_{i \in F} |z^*(x_{m_i})| < \sum_{i=0}^{n} \epsilon_i.$$

Without loss of generality, since (x_i) is bimonotone, we can assume that $z^*(x_i) = 0$, for all $i > \max\operatorname{supp}\xi^R_{k_{j_0}}$. Our previous observation yields that z^* is ξ-good for (λ, R, k_j), for all $j \le q$, and thus,

$$\sum_{i\in F} |z^*(x_{m_i})| + |z^*(x_{p_j})| \ge \sum_{i=0}^{n+1} \epsilon_i, \text{ for all } j \le q.$$

Hence, $|z^*(x_{p_j})| \ge \epsilon_{n+1}$, for all $j \le q$. We have reached a contradiction since (x_i) is weakly null and $q \in \mathbb{N}$ was arbitrary. Concluding, there exists $L \in [M_n]$ such that $[L] \subset \Delta_F$. By repeating the previous argument successively over all possible subsets of T_{n+1}, we obtain $M_{n+1} \in [M_n]$ such that $[M_{n+1}] \subset \Delta_F$, for all $F \subset T_{n+1}$. This completes the inductive construction. Set $M = (m_i)$. Let $L \in [M]$ and $k \in \mathbb{N}$ and suppose that there exists $x^* \in B_{X^*}$ which is ξ-good for (λ, L, k). Now let $m_n = \max\operatorname{supp}\xi^L_k$ and $F = \{i < n : m_i \notin \bigcup_{j=1}^k \operatorname{supp}\xi^L_j\}$. Our construction yields that $(m_i)_{i\in T_n}$ and $\{m_i : i \ge n+1\}$ satisfy $(F - E_n)$. We also have, by stability, that

$$\xi^L_j = \xi^{R_n}_j, \text{ for all } j \le k,$$

where $R_n = \{m_i : i \in T_n \setminus F\} \cup \{m_i : i > n+1\}$, and therefore there exists $y^* \in B_{X^*}$ which is ξ-good for (λ, L, k) and such that

$$\sum_{i\in F} |y^*(x_{m_i})| < \sum_{i=0}^{n} \epsilon_i < \epsilon.$$

But (x_i) is bimonotone and thus we can assume that $y^*(x_i) = 0$ for all $i > m_n$. Hence, $\sum_{i\in M\setminus\bigcup_{j=1}^k \operatorname{supp}\xi^L_j} |y^*(x_i)| < \epsilon$, as desired. □

Proof of Theorem 1.9. Let $\delta > 0$. Our goal is to find $M \in [\mathbb{N}]$ and a constant $C(\delta) > 0$ such that if $L \in [M]$, $n \in \mathbb{N}$, and $(a_i)_{i=1}^n$ are scalars in $[-1, 1]$, then

$$\left\|\sum_{i\in F} a_i(\xi^L_i \cdot s)\right\| \le C(\delta) \max\left\{\delta, \left\|\sum_{i=1}^n a_i(\xi^L_i \cdot s)\right\|\right\},$$

for all $F \subset \{i \le n : |a_i| \ge \delta\}$. If this is accomplished, then a simple diagonalization argument yields $M \in [\mathbb{N}]$ which works for all $\delta > 0$. We let $\lambda_k = 1 + \frac{k+1}{\delta}$, for all $k \in \mathbb{N} \cup \{0\}$. Inductively we construct a decreasing sequence $(M_k)_{k=0}^\infty$ of infinite subsets of $\mathbb{N}$ such that for all $k \in \mathbb{N} \cup \{0\}$, M_k satisfies the conclusion of Lemma 3.2 for "λ" $= \lambda_k$ and "ϵ" $= \delta$. Next choose $m_1 < m_2 < \cdots$ so that $m_i \in M_i$, for all $i \in \mathbb{N}$. Let $M = (m_i)$. We shall show that M is desired. To this end, let $L \in [M]$ and scalars $(a_i)_{i=1}^n$ in $[-1, 1]$. Let $F \subset \{1, \cdots, n\}$ such that $|a_i| \ge \delta$, for all $i \in F$. Choose $k \in \mathbb{N} \cup \{0\}$ so that

$$k \le \left\|\sum_{i=1}^n a_i(\xi^L_i \cdot s)\right\| < k + 1$$

and set $F_k = \{i \in F : i \ge k\}$. We claim that

$$\left\|\sum_{i\in F_k} a_i(\xi^L_i \cdot s)\right\| \le 4\lambda_k.$$

Assume this is not the case and choose $x^* \in B_{X^*}$ such that

$$\Big|\sum_{i\in F_k} a_i x^*(\xi_i^L \cdot s)\Big| > 4\lambda_k.$$

We can further choose $G_k \subset F_k$ such that

$$\left|\sum_{i\in G_k} a_i x^*(\xi_i^L \cdot s)\right| > \lambda_k,$$

the scalars $(a_i)_{i\in G_k}$ are all of the same sign, and, $x^*(\xi_i^L \cdot s) \geq 0$, for all $i \in G_k$, (by replacing x^* by $-x^*$, if necessary). Observe that $\operatorname{supp} \xi_i^L \subset M_k$, when $k \leq i \leq n$, and thus by stability, there exists $y^* \in B_{X^*}$ such that

$$y^*(\xi_i^L \cdot s) \geq 0, \text{ for all } i \in G_k,$$

$$\sum_{i\in G_k} y^*(\xi_i^L \cdot s) > \lambda_k \text{ and } \sum_{i\in M_k \setminus \bigcup_{j\in G_k} \operatorname{supp} \xi_j^L} |y^*(x_i)| < \delta.$$

But now,

$$\begin{aligned}\left|\sum_{i=k}^n a_i y^*(\xi_i^L \cdot s)\right| &\geq \left|\sum_{i\in G_k} a_i y^*(\xi_i^L \cdot s)\right| - \sum_{i\in\{k,\cdots,n\}\setminus G_k} |y^*(\xi_i^L \cdot s)| \\ &> \sum_{i\in G_k} |a_i| y^*(\xi_i^L \cdot s) - \sum_{i\in M_k \setminus \bigcup_{j\in G_k} \operatorname{supp} \xi_j^L} |y^*(x_i)| \\ &> \delta\lambda_k - \delta = k+1.\end{aligned}$$

Thus, $\| \sum_{i=k}^n a_i(\xi_i^L \cdot s)\| > k+1$ and since (x_n) is bimonotone, we also have that $\| \sum_{i=1}^n a_i(\xi_i^L \cdot s)\| > k+1$ which is a contradiction. Therefore, our claim holds and hence

$$\| \sum_{i\in F} a_i(\xi_i^L \cdot s)\| \leq \begin{cases} 4(1+\frac{1}{\delta}), & \text{if } \| \sum_{i=1}^n a_i(\xi_i^L \cdot s)\| < 1; \\ k(5+\frac{8}{\delta}), & \text{if } k \leq \| \sum_{i=1}^n a_i(\xi_i^L \cdot s)\| < k+1,\ k \in \mathbb{N}. \end{cases}$$

Concluding,

$$\left\|\sum_{i\in F} a_i(\xi_i^L \cdot s)\right\| \leq C(\delta) \max\left\{\delta, \left\|\sum_{i=1}^n a_i(\xi_i^L \cdot s)\right\|\right\},$$

where, $C(\delta) = \frac{1}{\delta}(5+\frac{8}{\delta})$. The proof of Theorem 1.9 is now complete. □

Remark. A refinement of the proof of Lemma 3.2, yields that given $\theta > 0$ and $\epsilon > 0$, there exists $M \in [\mathbb{N}]$ satisfying the following property: If $L \in [M]$, $k \in \mathbb{N}$ and there exists $x^* \in B_{X^*}$ which is ξ-good for (λ, L, k), then there exists $y^* \in B_{X^*}$ with $y^*(\xi_i^L \cdot s) \geq 0$ for $i \leq k$ and such that

$$\sum_{i=1}^k y^*(\xi_i^L \cdot s) \geq (1-\theta) \sum_{i=1}^k x^*(\xi_i^L \cdot s) \text{ and}$$

$$\sum_{n\in M\setminus \bigcup_{i=1}^k \operatorname{supp} \xi_i^L} |y^*(x_n)| < \epsilon \sum_{i=1}^k y^*(\xi_i^L \cdot s).$$

This property in turn implies that the constant $C(\delta)$, found in the proof of Theorem 1.9, is actually proportional to $\frac{1}{\delta}$.

Elton's aforementioned theorem follows immediately if we let $\xi = 0$ in the statement of Theorem 1.9. As a corollary to Elton's theorem one obtains the following dichotomy [E], [O1]:

Corollary 3.3. *A normalized weakly null sequence (x_n) in a Banach space, either has a subsequence equivalent to the unit vector basis of c_0, or, a Schauder basic subsequence (x_{m_n}) which is semi-boundedly complete. If the latter alternative occurs, then the sequence of the biorthogonal functionals to (x_{m_n}), converges weakly to zero in the dual of the closed linear span of (x_{m_n}).*

Our next result is the dual version of Lemma 3.2.

Lemma 3.4. *Let $s = (x_n)$ be a weakly null sequence in the Banach space X and $\xi < \omega_1$. Given $\epsilon > 0$, $\delta > 0$, $N \in [\mathbb{N}]$, there exists $M \in [N]$ satisfying the following property: If $k \in \mathbb{N}$, $L \in [M]$, and there exists $x^* \in B_{X^*}$ so that $x^*(\xi_n^L \cdot s) \geq \delta$, for all $n \leq k$, then there exists $y^* \in B_{X^*}$ so that $y^*(\xi_n^L \cdot s) \geq \delta$, for all $n \leq k$, and,*

$$\sum_{n \in M \setminus \bigcup_{i=1}^k \operatorname{supp} \xi_i^L} |y^*(x_n)| < \epsilon.$$

Proof. The proof is similar to that of Lemma 3.2. Let us now say that the functional $x^* \in B_{X^*}$ is good for (δ, L, k), if $x^*(\xi_n^L \cdot s) \geq \delta$, for all $n \leq k$. Next choose $(\epsilon_i)_{i=0}^\infty$, a sequence of positive scalars such that $\sum_{i=0}^\infty \epsilon_i < \epsilon$. Using the same notation and terminology as in Lemma 3.2, let $n \in \mathbb{N} \cup \{0\}$ and $F \subset T_n$. The n-tuple $(m_i)_{i \in T_n}$ and the infinite subset L of $\mathbb{N}$, $(L = (l_i))$, are said to satisfy property $(F - E_n^*)$, provided that $m_n < l_1$, if $n \geq 1$, and the following statement holds:

If $k \in \mathbb{N}$, and there exists $x^* \in B_{X^*}$ which is good for $(\lambda, \{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}, k)$, then there exists $y^* \in B_{X^*}$ which is good for $(\lambda, \{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}, k)$ and such that

$$\sum_{j \in F} |y^*(x_{m_j})| + |y^*(x_{l_1})| < \sum_{i=0}^n \epsilon_i.$$

Let us also say that $(m_i)_{i \in T_n}$ and L satisfy property (E_n^*), if they satisfy property $(F - E_n^*)$, for every $F \subset T_n$.

We shall inductively construct an increasing sequence $(m_n)_{n=1}^\infty$ of elements of N, and a decreasing sequence $(M_n)_{n=0}^\infty$ of infinite subsets of N with $m_n \in M_{n-1}$, if $n \geq 1$, so that for every $n \in \mathbb{N} \cup \{0\}$, if $L \in [M_n]$, then $(m_i)_{i \in T_n}$ and L satisfy property (E_n^*).

The first inductive step is similar to the general one and therefore we shall not discuss it. Assume that $(m_i)_{i \in T_n}$ and $M_0 \supset \cdots \supset M_n$, infinite subsets of N, with $m_i \in M_{i-1}$ for $1 \leq i \leq n$ have been constructed so that if $i \leq n$ and $L \in [M_i]$, then $(m_j)_{j \in T_i}$ and L satisfy property (E_i^*). Let $m_{n+1} = \min M_n$. Fix $F \subset T_{n+1}$ and define

$$\Delta_F = \{L \in [M_n], L = (l_i), : (m_i)_{i \in T_{n+1}} \text{ and } L \text{ satisfy } (F - E_{n+1}^*)\}.$$

Clearly, Δ_F is closed in $[M_n]$ and therefore Ramsey. Arguing as we did in the proof of Lemma 3.2, we obtain $M_{n+1} \in [M_n]$ such that $[M_{n+1}] \subset \Delta_F$, for every $F \subset T_{n+1}$. Indeed, we need only modify the definition of k_{j_0} in the argument of Lemma 3.2. We alternatively set $k_{j_0} = \max\{k_j : j \leq q\}$ and observe that if y^* is good for (δ, R, k_{j_0}), then y^* is also good for (δ, R, k_j), for every $j \leq q$. The argument of Lemma 3.2 is now carried over unaltered yielding the proof of Lemma 3.4. $\square$

In the proof of the Schreier unconditionality theorem [M-R], [O2], we shall make use of the following:

Lemma 3.5. *Let (x_n) be a weakly null sequence in the Banach space X. Let $\epsilon > 0$, $\delta > 0$ and $k \in \mathbb{N}$. There exists $Q \in [\mathbb{N}]$, $Q = (q_n)$, such that for every $x^* \in B_{X^*}$ and $F \in [\mathbb{N}]^k$ there exists $y^* \in B_{X^*}$ satisfying*

$$\sum_{i\in F} |x^*(x_{q_i}) - y^*(x_{q_i})| < \delta \quad \textit{and} \quad \sum_{i\notin F} |y^*(x_{q_i})| < \epsilon.$$

Proof. Once again, the proof is much similar to that of Lemma 3.2. First choose $(\epsilon_i)_{i=0}^{\infty}$, a sequence of positive scalars such that $\sum_{i=0}^{\infty} \epsilon_i < \epsilon$. Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ be an element of $[-1,1]^k$. We shall say that the functional $x^* \in B_{X^*}$ is $\vec{\lambda}$-good for L, where $L = (l_i)$ is an infinite subset of $\mathbb{N}$, if $\sum_{i=1}^{k} |x^*(x_{l_i}) - \lambda_i| < \delta$. Using the same notation and terminology as in Lemma 3.2, let $n \in \mathbb{N} \cup \{0\}$ and $F \subset T_n$. The n-tuple $(m_i)_{i\in T_n}$ and the infinite subset L of $\mathbb{N}$, $(L = (l_i))$, are said to satisfy property $(F - O_n)$, provided that $m_n < l_1$, if $n \geq 1$, and the following statement holds:

If there exists $x^* \in B_{X^*}$ which is $\vec{\lambda}$- good for $\{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}$, then there exists $y^* \in B_{X^*}$ which is $\vec{\lambda}$-good for $\{m_j : j \in T_n \setminus F\} \cup \{l_j : j \geq 2\}$ and such that

$$\sum_{j\in F} |y^*(x_{m_j})| + |y^*(x_{l_1})| < \sum_{i=0}^{n} \epsilon_i.$$

Let us also say that $(m_i)_{i\in T_n}$ and L satisfy property (O_n), if they satisfy property $(F - O_n)$, for every $F \subset T_n$. We shall inductively construct an increasing sequence $(m_n)_{n=1}^{\infty}$ of elements of $\mathbb{N}$, and a decreasing sequence $(M_n)_{n=0}^{\infty}$ of infinite subsets of $\mathbb{N}$ with $m_n \in M_{n-1}$, if $n \geq 1$, so that for every $n \in \mathbb{N} \cup \{0\}$, if $L \in [M_n]$, then $(m_i)_{i\in T_n}$ and L satisfy property (O_n).

The first inductive step is similar to the general one and therefore we shall not discuss it. Assume that $(m_i)_{i\in T_n}$ and $M_0 \supset \cdots \supset M_n$, infinite subsets of N, with $m_i \in M_{i-1}$ for $1 \leq i \leq n$ have been constructed so that if $i \leq n$ and $L \in [M_i]$, then $(m_j)_{j\in T_i}$ and L satisfy property (O_i). Let $m_{n+1} = \min M_n$. Fix $F \subset T_{n+1}$ and define

$$\Delta_F = \{L \in [M_n], L = (l_i), : (m_i)_{i\in T_{n+1}} \text{ and } L \text{ satisfy } (F - O_{n+1})\}.$$

Clearly, Δ_F is closed in $[M_n]$ and therefore Ramsey. Arguing as we did in the proofs of Lemmas 3.2 and 3.4 we obtain $M_{n+1} \in [M_n]$ such that $[M_{n+1}] \subset \Delta_F$, for every $F \subset T_{n+1}$. The inductive construction is now complete and we set $M = (m_i)$. It follows, by our construction, that if $F \in [\mathbb{N}]^k$ and $x^* \in B_{X^*}$ is $\vec{\lambda}$-good for $\{m_j : j \in F\} \cup \{m_j : j > \max F\}$, then there exists $y^* \in B_{X^*}$ $\vec{\lambda}$-good for $\{m_j : j \in F\} \cup \{m_j : j > \max F\}$ and such that $\sum_{j\notin F} |y^*(x_{m_j})| < \epsilon$. Let us then say that M works for $\vec{\lambda}$. Finally, let $\mathcal{E}$ be a finite δ-net in $[-1,1]^k$, and choose $Q \in [\mathbb{N}]$ which works for every $\vec{\lambda} \in \mathcal{E}$. It is easily verified that Q is desired. □

Lemma 3.4 provides an alternative proof of the fact that every normalized weakly null sequence admits a convexly unconditional subsequence.

Proof of Theorem 1.10. Let $\delta > 0$. It suffices to find $M \in [\mathbb{N}]$, $M = (m_i)$, so that if $F \in [\mathbb{N}]^{<\infty}$ and $(\lambda_i)_{i\in F}$ are scalars satisfying $\| \sum_{i\in F} \lambda_i x_{m_i} \| > \delta$ and $\sum_{i\in F} |\lambda_i| \leq 1$,

then $\|\sum_{i=1}^{\infty} a_i x_{m_i}\| > \frac{\delta^2}{32}$, for all choices of scalars $(a_i)_{i=1}^{\infty} \subset c_{00}$ with $\max_i |a_i| \leq 1$ and such that $|a_i| = |\lambda_i|$, for all $i \in F$. Once this is accomplished, a simple diagonalization argument yields $M \in [\mathbb{N}]$ which works for all $\delta > 0$. To this end, let $M = (m_i)$ be the infinite subset of $\mathbb{N}$ resulting from Lemma 3.4 applied on the sequence (x_n) for $\xi = 0$, "δ"$= \frac{\delta}{2}$ and "ϵ"$= \frac{\delta^2}{32}$. Now let $F \in [\mathbb{N}]^{<\infty}$ and $(\lambda_i)_{i\in F}$ be scalars such that $\|\sum_{i\in F} \lambda_i x_{m_i}\| > \delta$ and $\sum_{i\in F} |\lambda_i| \leq 1$. Choose $x^* \in B_{X^*}$ such that $|\sum_{i\in F} \lambda_i x^*(x_{m_i})| > \delta$ and set

$$G = \left\{ i \in F : |x^*(x_{m_i})| \geq \frac{\delta}{2} \right\}.$$

Clearly, $|\sum_{i\in G} \lambda_i x^*(x_{m_i})| > \frac{\delta}{2}$. Next, let $(a_i)_{i=1}^{\infty} \subset c_{00}$ with $\max_i |a_i| \leq 1$ and $|a_i| = |\lambda_i|$, for all $i \in F$. By splitting G into four sets in the obvious manner, we find $H \subset G$ such that

$$\left| \sum_{i\in H} \lambda_i x^*(x_{m_i}) \right| > \frac{\delta}{8},$$

the scalars $(\lambda_i)_{i\in H}$ are all of the same sign, and, $x^*(x_{m_i}) \geq \frac{\delta}{2}$ for all $i \in H$, (by replacing x^* by $-x^*$, if necessary). Now choose $y^* \in B_{X^*}$ with $y^*(x_{m_i}) \geq \frac{\delta}{2}$, for all $i \in H$, and such that $\sum_{i\notin H} |y^*(x_{m_i})| < \frac{\delta^2}{32}$. It follows that

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} a_i x_{m_i} \right\| &\geq \left| \sum_{i\in H} a_i y^*(x_{m_i}) \right| - \sum_{i\notin H} |y^*(x_{m_i})| \\ &> \sum_{i\in H} |\lambda_i| \frac{\delta}{2} - \frac{\delta^2}{32} > \frac{\delta^2}{32}. \end{aligned}$$

□

Corollary 3.6. *For a normalized weakly null sequence $s = (x_n)$ and $\xi < \omega_1$, the following are equivalent:*

1. *There exists $M \in [\mathbb{N}]$, $M = (m_i)$, so that (x_{m_i}) is an l_1^{ξ} spreading model.*
2. *There exist $N \in [\mathbb{N}]$ and $\delta > 0$ such that $S_\xi(N) \subset \mathfrak{F}_\delta(s)$.*

Proof. Suppose first that 1 holds and let $C > 0$ such that

$$\left\| \sum_{i\in F} \lambda_i x_{m_i} \right\| \geq C \sum_{i\in F} |\lambda_i| \quad \text{for all } F \in S_\xi, \qquad \text{and scalars } (\lambda_i)_{i\in F}.$$

Let t denote the sequence (y_i), where $y_i = x_{m_i}$, for all $i \in \mathbb{N}$. Then, $\|\xi_1^L \cdot t\| \geq C$, for all $L \in [\mathbb{N}]$. It follows that for $\delta = \frac{C}{2}$, the hereditary family

$$\{F \in [\mathbb{N}]^{<\infty} : M(F) \in \mathfrak{F}_\delta(s)\},$$

is $(1, \xi, \mathbb{N}, \delta)$ large. Corollary 2.17 now yields $N \in [M]$ so that $S_\xi(N) \subset \mathfrak{F}_\delta(s)$ and thus 2 holds.

Conversely, assume that 2 holds and choose $M = (m_i) \in [N]$ as in the proof of Theorem 1.10, applied on the sequence $(x_n)_{n\in N}$ for "δ"$= \frac{\delta}{4}$. Let $F \in S_\xi$ and scalars $(\lambda_i)_{i\in F}$ such that $\sum_{i\in F} |\lambda_i| = 1$. We claim that $\|\sum_{i\in F} \lambda_i x_{m_i}\| \geq \frac{\delta^2}{512}$,

which evidently yields 1. Indeed, by our assumption, there exists $x^* \in B_{X^*}$ such that $|x^*(x_{m_i})| \geq \delta$, for all $i \in F$. Next choose $G \subset F$ such that

$$\sum_{i \in G} |\lambda_i| \geq \frac{1}{4},$$

the scalars $(\lambda_i)_{i \in G}$ are all of the same sign, and, $x^*(x_{m_i}) \geq \delta$, for all $i \in G$, (by replacing x^* by $-x^*$ if necessary). Therefore, $\| \sum_{i \in G} \lambda_i x_{m_i} \| \geq | \sum_{i \in G} \lambda_i x^*(x_{m_i})| \geq \frac{\delta}{4}$, and hence applying Theorem 1.10 we obtain that $\| \sum_{i \in F} \lambda_i x_{m_i} \| \geq \frac{1}{32}(\frac{\delta}{4})^2 = \frac{\delta^2}{512}$, as claimed. □

Proof of Theorem 1.7. Assume that 2 does not hold. Let $M \in [\mathbb{N}]$ and $\epsilon > 0$. It is easily seen that the set

$$\mathcal{A}_\epsilon = \{L \in [M] : \|\xi_1^L \cdot s\| < \epsilon\}$$

where $s = (x_n)$, is closed in $[M]$ and therefore Ramsey. If it were the case that $[P] \bigcap \mathcal{A}_\epsilon = \emptyset$, for some $P \in [M]$, then the family $\mathfrak{F}_{\frac{\epsilon}{2}}(s)$ would be $(1, \xi, P, \frac{\epsilon}{2})$ large, and hence, by Corollaries 2.17 and 3.6, $(x_n)_{n \in L}$ would in turn be an ℓ_1^ξ spreading model, for some $L \in [P]$ contradicting our assumption. It follows now that we can construct (M_n), a decreasing sequence of infinite subsets of $\mathbb{N}$, such that for all $n \in \mathbb{N}$,

$$\|\xi_1^L \cdot s\| < \frac{1}{n}, \text{ for all } L \in [M_n].$$

Now let N be any infinite subset of M almost contained in each M_n, and it is easy to verify that (x_n) is (L, ξ) convergent, for all $L \in [N]$.

In order to show that 1 and 2 are mutually exclusive, assume that (x_{m_i}) is an ℓ_1^ξ spreading model with constant C. Let $0 < \epsilon < \frac{C}{1+C}$. We can choose $N \in [M]$ so that

$$F \setminus \{\min F\} \in S_\xi(M), \quad \text{for all } F \in S_\xi[N], \quad \text{(by Lemma 2.4)},$$
$$\|\xi_1^L\|_0 < \epsilon, \quad \text{for all } L \in [N], \quad \text{(by Proposition 2.15)}.$$

It follows now that for every $L \in [N]$,

$$\|\xi_n^L \cdot s\| > C - (1 + C)\epsilon > 0, \quad \text{for all } n \in \mathbb{N},$$

and thus (x_n) is not (L, ξ) convergent. Hence 1 does not hold. □

An immediate consequence of Theorem 1.7 is Corollary 1.8.

Proof of Corollary 1.8. Suppose that $s = (x_n)$ is an ℓ_1^ξ spreading model with constant C. It follows that $\|\xi_n^L \cdot s\| \geq C$, for all $n \in \mathbb{N}$ and $L \in [\mathbb{N}]$. Next, choose according to Theorem 1.7, $N \in [\mathbb{N}]$ so that (x_n) is $(L, \xi + 1)$ convergent, for every $L \in [N]$. Evidently, $(\xi_n^L \cdot s)$ is Cesaro summable for all $L \in [N]$. □

We continue our applications of Theorem 1.10 with the following.

Proof of Theorem 1.11. Let C be the ℓ_1^ξ spreading model constant of (x_n). Let $M \in [\mathbb{N}]$, $M = (m_i)$ satisfying the conclusion of Theorem 1.10 for the sequence (x_n) and "δ"$= \frac{C}{2}$. We claim that (x_{m_i}) is S_ξ unconditional. Indeed, let $n \in \mathbb{N}$ and scalars $(a_i)_{i=1}^n$ be given. Let $F \subset \{1, \cdots, n\}$, $F \in S_\xi$, such that $\| \sum_{i \in F} a_i x_{m_i} \| = 1$. It follows that $C \sum_{i \in F} |a_i| \leq 1$ and $\| \sum_{i \in F} C a_i x_{m_i} \| > \frac{C}{2}$. If there exists $j \leq n$ so that $C|a_j| > 1$, then, since (x_n) is bimonotone, we obtain that $\| \sum_{i=1}^n a_i x_{m_i} \| > \frac{1}{C}$.

So assuming that $C|a_i| \leq 1$, for all $i \leq n$, we obtain through Theorem 1.10 that $\|\sum_{i=1}^n a_i x_{m_i}\| > \frac{C}{128}$. Hence,

$$\Big\|\sum_{i\in F} a_i x_{m_i}\Big\| \leq \frac{128}{C}\Big\|\sum_{i=1}^n a_i x_{m_i}\Big\|,$$

for every $F \in S_\xi$ and all choices of scalars $(a_i)_{i=1}^n$. □

We also obtain the result on Schreier unconditionality [M-R], [O2].

Corollary 3.7. *Let (x_n) be a normalized weakly null sequence in X and $\epsilon > 0$. There exists a subsequence (x_{m_i}) of (x_n) which is $2+\epsilon$ S_1 unconditional.*

Proof. First choose $\theta > 0$ such that $(1+\theta)(2+\theta) < 2+\epsilon$. By passing to a subsequence, if necessary, we can assume that (x_n) is Schauder basic with basis constant $1+\theta$. We first show that for every $k \in \mathbb{N}$ and $N \in [\mathbb{N}]$, there exists $L \in [N]$, $L = (l_i)$ so that

$$\Big\|\sum_{i\in F} a_i x_{l_i}\Big\| \leq (1+\theta)\Big\|\sum_{i=1}^\infty a_i x_{l_i}\Big\|,$$

for every $F \in [\mathbb{N}]^k$ and all choices of scalars (a_i) in c_{00}. Indeed, apply Lemma 3.5 to the sequence $(x_n)_{n\in N}$ to obtain $L \in [N]$, $L = (l_i)$, satisfying the conclusion of that lemma for k, "δ"$= \frac{\theta}{6}$, and "ϵ"$= \frac{\theta}{6}$. Now let $(a_i) \subset c_{00}$ such that $\|\sum_{i=1}^\infty a_i x_{l_i}\| = 1$. Let $F \in [\mathbb{N}]^k$ and choose $x^* \in B_{X^*}$ such that

$$\Big\|\sum_{i\in F} a_i x_{l_i}\Big\| = \Big|\sum_{i\in F} a_i x^*(x_{l_i})\Big|.$$

Then choose $y^* \in B_{X^*}$ such that

$$\sum_{i\in F} |x^*(x_{l_i}) - y^*(x_{l_i})| < \frac{\theta}{6} \quad \text{and} \quad \sum_{i\notin F} |y^*(x_{l_i})| < \frac{\theta}{6}.$$

We now have the estimate

$$\begin{aligned} 1 =& \Big\|\sum_{i=1}^\infty a_i x_{l_i}\Big\| \geq \Big|\sum_{i=1}^\infty a_i y^*(x_{l_i})\Big| \\ &\geq \Big|\sum_{i\in F} a_i x^*(x_{l_i})\Big| - \sum_{i\in F} |a_i||x^*(x_{l_i}) - y^*(x_{l_i})| - \sum_{i\notin F} |a_i||y^*(x_{l_i})| \\ &\geq \Big\|\sum_{i\in F} a_i x_{l_i}\Big\| - 3\frac{\theta}{6} - 3\frac{\theta}{6}, \end{aligned}$$

and thus, $\|\sum_{i\in F} a_i x_{l_i}\| \leq (1+\theta)\|\sum_{i=1}^\infty a_i x_{l_i}\|$, as desired. We can now choose a decreasing sequence (M_k) of infinite subsets of $\mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$\Big\|\sum_{m\in F} a_m x_m\Big\| \leq (1+\theta)\Big\|\sum_{m\in M_k} a_m x_m\Big\|,$$

for all $F \in [M_k]^j$ and $j \leq k$, and all choices of scalars $(a_m)_{m\in M_k} \subset c_{00}$. Finally, choose $m_1 < m_2 < \cdots$ with $m_i \in M_i$, for all $i \in \mathbb{N}$, and set $M = (m_i)$. It is easily verified that the subsequence (x_{m_i}) is $2+\epsilon$ S_1 unconditional. □

The final results of this section concern the duality between ${c_0}^\xi$ and ℓ_1^ξ spreading models. We first recall the following:

Definition 3.8. A sequence (x_n) in a Banach space is called a ${c_0}^\xi$ spreading model, $1 \le \xi < \omega_1$, if there exists a constant $C > 0$ such that

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \le C \max_{i\in F} |a_i|,$$

for every $F \in S_\xi$ and all choices of scalars $(a_i)_{i\in F}$.

Proof of Theorem 1.12. Once again, we assume that our sequence (x_n) is bimonotone. We can also assume, without loss of generality, that

$$\sum_{i=1}^{\infty}\sum_{j\neq i} |x_i^*(x_j)| < \frac{\epsilon}{2}.$$

Furthermore, we shall assume that (x_n) satisfies the conclusion of Theorem 1.9 for $\xi = 0$ and $M = \mathbb{N}$. That is, for all $\delta > 0$ there exists a constant $C(\delta) > 0$ such that for every $n \in \mathbb{N}$ and all scalars $(a_i)_{i=1}^n$ in $[-1, 1]$, if $F \subset \{1, \cdots, n\}$ and $|a_i| \ge \delta$ for all $i \in F$, then

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \le C(\delta)\Big\|\sum_{i=1}^{n} a_i x_i\Big\|.$$

Assume first that 2 holds and choose $L \in [M]$, $L = (l_i)$, and $C > 0$ so that

$$\Big\|\sum_{i\in F} a_i x_{l_i}\Big\| \le C \max_{i\in F} |a_i|,$$

for every $F \in S_\xi$ and all choices of scalars $(a_i)_{i\in F}$. We claim that $(x_{l_i}^*|X_M)$ is an ℓ_1^ξ spreading model. Indeed, let $F \in S_\xi$ and scalars $(a_i)_{i=1}^n$ be given. For each $i \in F$, let ϵ_i be the sign of $a_i x_{l_i}^*(x_{l_i})$. Of course, $\|\sum_{i\in F} \epsilon_i x_{l_i}\| \le C$. Therefore,

$$\begin{aligned} C\Big\|\sum_{i\in F} a_i x_{l_i}^*|X_M\Big\| &\ge \Big|\sum_{i\in F} a_i x_{l_i}^*(\sum_{j\in F} \epsilon_j x_{l_j})\Big| \\ &\ge \sum_{i\in F} |a_i||x_{l_i}^*(x_{l_i})| - \sum_{i\in F} |a_i| \sum_{j\neq i} |x_{l_i}^*(x_{l_j})| \\ &\ge \frac{\epsilon}{2}\sum_{i\in F} |a_i| \end{aligned}$$

and thus 1 holds.

Conversely, assume that 1 holds. Choose, according to Corollary 3.6, $L \in [M]$, $L = (l_i)$ and $\delta > 0$ such that

$$S_\xi(L) \subset \{F \in [\mathbb{N}]^{<\infty} : \exists\, x^{**} \in B_{X_M^{**}} \text{ with } |x^{**}(x_n^*|X_M^*)| \ge \delta, \forall n \in F\}.$$

It follows that

$$S_\xi(L) \subset \{F \in [\mathbb{N}]^{<\infty} : \exists\, x \in B_{X_M} \text{ with } |x_n^*(x)| > \frac{\delta}{2}, \forall n \in F\}.$$

We now claim that for every $x \in X_M$, the sequence $\big(\frac{x_{m_i}^*(x)}{x_{m_i}^*(x_{m_i})} x_{m_i}\big)$, where $M = (m_i)$, is series bounded. Indeed, let $x = \sum_{i=1}^\infty c_i x_{m_i} \in B_{X_M}$ and note that by

monotonicity we have that $|c_i| \leq 1$, for all $i \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$,

$$\left\|\sum_{i=1}^{k} \frac{x^*_{m_i}(x)}{x^*_{m_i}(x_{m_i})} x_{m_i}\right\| \leq \left\|\sum_{i=1}^{k} c_i x_{m_i}\right\| + \left\|\sum_{i=1}^{k} \frac{1}{x^*_{m_i}(x_{m_i})}\Big(\sum_{j\neq i} c_j x^*_{m_i}(x_{m_j})\Big) x_{m_i}\right\|$$
$$\leq \|x\| + \sum_{i=1}^{k} \frac{1}{\epsilon} \sum_{j\neq i} |x^*_{m_i}(x_{m_j})| < \frac{3}{2}$$

and our claim holds. Now let $F \in S_\xi$ and choose $x \in B_{X_M}$ such that $|x^*_{l_i}(x)| > \frac{\delta}{2}$, for all $i \in F$. Since $\big|\frac{x^*_{l_i}(x)}{x^*_{l_i}(x_{l_i})}\big| > \frac{\delta}{2}$, for all $i \in F$, our initial assumptions on the sequence (x_n) yield that

$$\left\|\sum_{i\in F} \frac{x^*_{l_i}(x)}{x^*_{l_i}(x_{l_i})} x_{l_i}\right\| \leq C\big(\frac{\delta\epsilon}{2}\big)\left\|\sum_{i=1}^{n} \frac{x^*_{m_i}(x)}{x^*_{m_i}(x_{m_i})} x_{m_i}\right\|,$$

where $m_n = l_{\max F}$. Thus, letting $b_i^F = \frac{x^*_{l_i}(x)}{x^*_{l_i}(x_{l_i})}$, for $i \in F$, we have that

$$\left\|\sum_{i\in F} b_i^F x_{l_i}\right\| \leq \frac{3}{2} C\left(\frac{\delta\epsilon}{2}\right)$$

and $\frac{\delta}{2} \leq |b_i^F| \leq \frac{1}{\epsilon}$, for all $i \in F$ and $F \in S_\xi$. A standard argument now shows that

$$\left\|\sum_{i\in F} a_i x_{l_i}\right\| \leq \frac{6}{\delta} C\left(\frac{\delta\epsilon}{2}\right) \max_{i\in F} |a_i|,$$

for all $F \in S_\xi$ and scalars $(a_i)_{i\in F}$. Hence, 1 implies 2. □

Definition 3.9. The Banach space X satisfies the ξ Dunford-Pettis property (ξ-DP), $1 \leq \xi < \omega_1$, if for every pair of weakly null sequences (x_n) and (x_n^*) in X and X^* respectively, with (x_n^*) ξ-convergent, we have that $\lim_n x_n^*(x_n) = 0$.

X is said to be hereditarily ξ-DP, if every subspace of X satisfies the ξ-DP.

Proof of Corollary 1.13. Assume first that 1 holds. Let Y be a subspace of X and consider the normalized weakly null sequences (x_n) and (x_n^*) in Y and Y^* respectively, with (x_n^*) ξ-convergent. Suppose that for some $\epsilon > 0$ and $M = (m_i) \in [\mathbb{N}]$, it was the case that $x^*_{m_i}(x_{m_i}) > \epsilon$, for all $i \in \mathbb{N}$. It follows that condition 2 of Theorem 1.12 is satisfied and therefore $(x^*_{m_i})$ admits a subsequence which is an ℓ_1^ξ spreading model in Y^*. This contradicts with Theorem 1.7, as (x_n^*) is ξ-convergent. Hence, Y satisfies the ξ-DP and 2 holds.

Conversely, assume that 2 holds. Let (x_n) be a normalized weakly null sequence in X admitting no subsequence which is a c_0^ξ spreading model. In particular, no subsequence of (x_n) is equivalent to the unit vector basis of c_0, and thus by Corollary 3.3 there exists $M = (m_i) \in [\mathbb{N}]$ such that the sequence $(x^*_{m_i}|X_M)$ is weakly null in X_M^* (we let (x_n^*) denote the sequence of the biorthogonal functionals of (x_n)). Our assumption further yields that condition 2 of Theorem 1.12, fails for the space X_M and the weakly null sequences (x_{m_i}) and $(x^*_{m_i})|X_M$ in X_M and X_M^* respectively. Thus condition 1 fails as well and so there exists $P = (p_i) \in [M]$ so that $x^*_{p_i}|X_P$ is ξ-convergent in X_P^* according to Theorem 1.7. But X_P is ξ-DP and thus, $1 = \lim_i x^*_{p_i}(x_{p_i}) = 0$, which is absurd. □

Remark. Corollary 1.13 can be viewed as a partial generalization of Cembranos theorem [C], [K-O], that a Banach space X is hereditarily Dunford-Pettis if, and only if, every normalized weakly null sequence in X admits a subsequence equivalent to the unit vector basis of c_0.

4. Boundedly convexly complete sequences

This section is devoted to the proof of Theorem 1.5 which immediately yields Theorem 1.4. Our interest is concentrated in weakly null sequences without boundedly convexly complete subsequences. In the next series of lemmas, we describe their structure. We remark here that for the Schreier spaces X_ξ, $\xi < \omega$, described in Section 1, it can be shown that they contain no boundedly convexly complete sequences. However, an example of a weakly null sequence (y_n) in X_ω, is given in [A-O], such that no convex block subsequence of (y_n) satisfies the weak Banach-Saks property. It turns out that some subsequence of (y_n), is boundedly convexly complete. We also mention that examples of boundedly convexly complete sequences can be constructed in $C(\omega^\omega)$, the Banach space of functions continuous on the ordinal interval $[1, \omega^\omega]$ endowed by the order topology.

In the sequel, $s = (x_n)$ will denote a normalized, bimonotone weakly null sequence in the Banach space X. We shall assume, without loss of generality, that s satisfies the conclusion of Theorem 1.9, for $M = \mathbb{N}$ and $\xi = 0$. That is, for every $\delta > 0$, there exists $C(\delta) > 0$ such that if $(a_i)_{i=1}^n$ are scalars in $[-1, 1]$, $n \in \mathbb{N}$, and $F \subset \{1, \cdots, n\}$ with $|a_i| \geq \delta$, for all $i \in F$, then $\| \sum_{i \in F} a_i x_i \| \leq C(\delta) \| \sum_{i=1}^n a_i x_i \|$.

Notation.
1. Given $t = (y_i)$, a sequence in a Banach space and $\mathbf{a} = (a_i)$, a scalar sequence, we let $\mathbf{a} \cdot t$ denote the sequence $(a_i y_i)$.
2. If $M \in [\mathbb{N}]$, $M = (m_i)$, we let t_M denote the sequence (y_{m_i}).
3. Let $P = (p_i)$ and $Q = (q_i)$ be infinite subsets of $\mathbb{N}$ with P almost contained in Q. Then $\mathbf{a}_{Q|P} = (c_i)$ is the scalar sequence obtained in the following manner: Set $c_i = 0$, if $q_i \notin P$. Then set $c_i = a_j$, if $q_i = p_j$, for some $j \in \mathbb{N}$.

Lemma 4.1. *Assume that $s = (x_n)$ has no subsequence which is b.c.c. There exist $M_0 \in [\mathbb{N}]$ and $\delta_0 > 0$ such that for every $L \in [M_0]$ there exist a sequence of scalars $(a_m)_{m \in L}$ with $(a_m x_m)_{m \in L}$ series bounded by 1, and a sequence $(F_i)_{i \in \mathbb{N}}$ of consecutive subsets of L so that the following are satisfied for every $i \in \mathbb{N}$:*

$$a_m \geq 0, \ \textit{for all } m \in F_i, \ \sum_{m \in F_i} a_m \leq 1, \ \textit{and} \ \left\| \sum_{m \in F_i} a_m x_m \right\| > \delta_0.$$

Proof. We first observe that if $P = (p_i)$ and $Q = (q_i)$ are infinite subsets of $\mathbb{N}$ with P almost contained in Q, and $\mathbf{a}$ is a scalar sequence, then

$$b(\mathbf{a}_{Q|P} \cdot s_Q) \leq b(\mathbf{a} \cdot s_P), \ \text{and } \tau(\mathbf{a}, s_P) \leq \tau(\mathbf{a}_{Q|P}, s_Q).$$

To prove the lemma, it suffices to find $M_0 \in [\mathbb{N}]$, $\delta_1 > 0$ and $1 \leq K < \infty$, so that for every $L \in [M_0]$ there exists a scalar sequence $\mathbf{a}$ with $b(\mathbf{a} \cdot s_L) \leq K$ and $\tau(\mathbf{a}, s_L) > \delta_1$. Once this is accomplished, then M_0 and $\delta_0 = \frac{\delta_1}{2K}$ satisfy the conclusion of the lemma.

We now claim that there exist $N \in [\mathbb{N}]$ and $\delta_1 > 0$ such that for all $L \in [N]$ there exists a scalar sequence $\mathbf{a}$ with

$$b(\mathbf{a} \cdot s_L) < \infty, \ \text{and } \tau(\mathbf{a}, s_L) > \delta_1.$$

If this is not the case, we construct a decreasing sequence (M_i) consisting of infinite subsets of $\mathbb{N}$ so that $\tau(\mathbf{a}, s_{M_i}) \leq \frac{1}{i}$, for every scalar sequence $\mathbf{a}$ with $b(\mathbf{a} \cdot s_{M_i}) < \infty$. Now let $M \in [\mathbb{N}]$ be almost contained in M_i, for all $i \in \mathbb{N}$, and choose a scalar sequence $\mathbf{a}$ such that

$$b(\mathbf{a} \cdot s_M) < \infty \quad \text{and} \quad \tau(\mathbf{a}, s_M) > 0.$$

It follows that

$$b(\mathbf{a}_{M_i|M} \cdot s_{M_i}) \leq b(\mathbf{a} \cdot s_M) < \infty,$$

for all $i \in \mathbb{N}$ and thus,

$$\tau(\mathbf{a}, s_M) \leq \tau(\mathbf{a}_{M_i|M}, s_{M_i}) \leq \frac{1}{i},$$

for all $i \in \mathbb{N}$, which is absurd. Therefore our claim holds.

We next claim that there exist $M_0 \in [N]$ and $K < \infty$ so that for every $L \in [M_0]$, there exists a scalar sequence $\mathbf{a}$ such that $b(\mathbf{a} \cdot s_L) \leq K$ and $\tau(\mathbf{a}, s_L) > \delta_1$. Were this claim false, we would choose a decreasing sequence (N_i) of infinite subsets of N such that for all $i \in \mathbb{N}$, if $\tau(\mathbf{a}, s_{N_i}) > \delta_1$, for some scalar sequence $\mathbf{a}$, then $b(\mathbf{a} \cdot s_{N_i}) > i$. Now let $T \in [N]$ be almost contained in each N_i and choose a scalar sequence $\mathbf{a}$ such that

$$b(\mathbf{a} \cdot s_T) < \infty \quad \text{and} \quad \tau(\mathbf{a}, s_T) > \delta_1.$$

It follows that

$$\tau(\mathbf{a}_{N_i|T}, s_{N_i}) \geq \tau(\mathbf{a}, s_T) > \delta_1,$$

for all $i \in \mathbb{N}$, and hence

$$b(\mathbf{a} \cdot s_T) \geq b(\mathbf{a}_{N_i|T} \cdot s_{N_i}) > i,$$

for all $i \in \mathbb{N}$ which is absurd. □

Remark. If $L \in [M_0]$, $(F_i)_{i=1}^{\infty}$ and $(a_m)_{m \in L}$, are as in the conclusion of Lemma 4.1, then we shall call the sequence $(\sum_{m \in F_i} a_m x_m)_{i \in \mathbb{N}}$, a sub-convex block subsequence of (x_i) supported by L and satisfying the conclusion of Lemma 4.1.

Lemma 4.2. *Let $s = (x_n)$ be a weakly null sequence having no subsequence which is b.c.c. Let $M_0 \in [\mathbb{N}]$ and $\delta_0 > 0$ satisfy the conclusion of Lemma 4.1 applied on s. Assume that $\alpha < \omega_1$ is a limit ordinal and denote by $(\alpha_n + 1)$ the sequence of ordinals associated to α. Suppose that for every $n \in \mathbb{N}$ and every $N \in [M_0]$ there exists $M \in [N]$ such that $(x_m)_{m \in M}$ is an $l_1^{\alpha_n + 1}$ spreading model. Then, for every $N \in [M_0]$, there exists $M \in [N]$ such that $(x_m)_{m \in M}$ is an l_1^{α} spreading model.*

Proof. Let $\mathcal{A}$ denote the set of those sub-convex combinations of the sequence s which are members of a (not necessarily the same) sub-convex block subsequence of s that satisfies the conclusion of Lemma 4.1.

We shall apply Corollary 2.12 to the sequence s and the family $\mathcal{A}$ in order to obtain $T \in [N]$ so that $S_\alpha(T) \subset \mathfrak{F}_{\frac{\delta_0}{2}}(s)$. Corollary 3.6 will then yield that for some $M \in [T]$, $(x_m)_{m \in M}$ is an l_1^α spreading model. To this end, it suffices to show that for every $i_0 \in \mathbb{N}$, every $\epsilon > 0$ and $L \in [N]$, there exists $x \in \mathcal{A}$ supported by L and such that $\|x\|_{\alpha_{i_0}} < \epsilon$. Suppose this is not the case and choose according to the hypothesis $P \in [L]$ so that $(x_m)_{m \in P}$ is an $l_1^{\alpha_{i_0}+1}$ spreading model with constant K. Without loss of generality, by Theorem 1.11, we can assume that $(x_m)_{m \in P}$ is

$S_{\alpha_{i_0}+1}$ unconditional with constant C. Next, choose by Lemma 2.4, $Q \in [P]$ such that if $F \in S_{\alpha_{i_0}+1}[Q]$, then $F \setminus \{\min F\} \in S_{\alpha_{i_0}+1}(P)$.

Since s has no b.c.c. subsequence there exist, by Lemma 4.1, a sequence of scalars $(a_q)_{q\in Q}$ and a sequence $(F_i)_{i=1}^{\infty}$ of consecutive subsets of Q with $(a_q x_q)_{q\in Q}$ series bounded by 1, such that for all $i \in \mathbb{N}$,

$$a_q \geq 0, \quad \text{for all } q \in F_i, \sum_{q\in F_i} a_q \leq 1 \quad \text{and} \quad \|\sum_{q\in F_i} a_q x_q\| > \delta_0.$$

Note that $\sum_{q\in F_i} a_q x_q \in \mathcal{A}$, and moreover, it is supported by L for all $i \in \mathbb{N}$. Thus, there exists, for every $i \in \mathbb{N}$,

$$G_i \subset F_i,\, G_i \in S_{\alpha_{i_0}} \quad \text{so that} \quad \sum_{q\in G_i} a_q \geq \epsilon.$$

Next choose $p \in \mathbb{N}$ such that $(p-1)K\epsilon > C$. Choose also $j \in \mathbb{N}$ with $p < \min F_j$. Then

$$\bigcup_{l=j+1}^{j+p} G_l \in S_{\alpha_{i_0}+1}[Q] \quad \text{and thus} \quad \bigcup_{l=j+2}^{j+p} G_l \in S_{\alpha_{i_0}+1}(P).$$

Therefore,

$$\Big\| \sum_{q\in\bigcup_{l=j+2}^{j+p} G_l} a_q x_q \Big\| \geq K \sum_{q\in\bigcup_{l=j+2}^{j+p} G_l} a_q \geq K(p-1)\epsilon > C.$$

However,

$$\Big\| \sum_{q\in\bigcup_{l=j+2}^{j+p} G_l} a_q x_q \Big\| \leq C \Big\| \sum_{q\leq \max G_{j+p}} a_q x_q \Big\| \leq C.$$

This contradiction completes the proof of the lemma. □

Lemma 4.3. *Let $s = (x_n)$ be a weakly null sequence having no subsequence which is b.c.c. There exist $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $\delta > 0$ so that the following are satisfied:*

1. *$(x_m)_{m\in M}$ is an ℓ_1^{ξ} spreading model yet no subsequence of $(x_m)_{m\in M}$ is an $\ell_1^{\xi+1}$ spreading model.*
2. *For every $N \in [M]$, there exist $L \in [N]$ and a sequence of scalars $(a_m)_{m\in N}$ with $(a_m x_m)_{m\in N}$ series bounded by 1 so that for all $i \in \mathbb{N}$*

$$a_m \geq 0, \text{ for all } m \in F_i^{\xi}(L), \sum_{m\in F_i^{\xi}(L)} a_m \leq 1 \quad \text{and} \quad \Big\| \sum_{m\in F_i^{\xi}(L)} a_m x_m \Big\| > \delta.$$

Proof. Let $M_0 \in [\mathbb{N}]$ and $\delta_0 > 0$ satisfy the conclusion of Lemma 4.1 applied on s. We define

$$\zeta = \min\{\alpha < \omega_1 : \exists\, P \in [M_0] \text{ such that } \forall L \in [P], \\ (x_n)_{n\in L} \text{ is not an } \ell_1^{\alpha} \text{ spreading model}\}.$$

It follows by the results in [A-A] that ζ is well defined. Moreover, Lemma 4.2 yields that ζ is not a limit ordinal and thus $\zeta = \xi + 1$, for some countable ordinal ξ. Now choose $P \in [M_0]$ so that no subsequence of $(x_n)_{n\in P}$ is an $\ell_1^{\xi+1}$ spreading model. Since $\xi < \zeta$, there exists $Q \in [P]$ so that $(x_n)_{n\in Q}$ is an ℓ_1^{ξ} spreading model and of course no subsequence of $(x_n)_{n\in Q}$ is an $\ell_1^{\xi+1}$ spreading model.

It follows now by Corollary 3.6, that for every $L \in [Q]$, the family $S_{\xi+1}(L)$ is not contained in $\mathfrak{F}_{\frac{\delta_0}{2}}(s)$. Using Lemma 2.9 as we did in the proof of Theorem 2.11, we find $M \in [Q]$ and $n_0 \in \mathbb{N}$, so that

$$\bigcup_{i=1}^{n_0} F_i^{\xi}(L) \notin \mathfrak{F}_{\frac{\delta_0}{2}}(s)$$

for all $L \in [M]$. It follows now, by Lemma 2.6, that every member of the family $\mathfrak{F}_{\frac{\delta_0}{2}}(s)[M]$ can be expressed as the union of at most n_0 consecutive S_ξ sets.

We are going to show that M and $\delta = \frac{\delta_0}{2n_0}$ satisfy 2. Indeed, let $N \in [M]$ and choose

$$n_1 < A_{n_1} < n_2 < A_{n_2} < \cdots$$

so that for all $i \in \mathbb{N}$, $n_i \in N$ and A_{n_i} is a maximal S_ξ subset of N. Set $N_1 = \{n_i : i \in \mathbb{N}\}$. By Lemma 4.1, there exist a sequence of scalars $(a_m)_{m\in N_1}$ and a sequence of consecutive subsets of N_1, $(F_i)_{i=1}^{\infty}$ with $(a_m x_m)_{m\in N_1}$ series bounded by 1, so that for every $i \in \mathbb{N}$

$$a_m \geq 0, \quad \text{for all } m \in F_i, \sum_{m\in F_i} a_m \leq 1, \text{ and } \left\| \sum_{m\in F_i} a_m x_m \right\| > \delta_0.$$

We next choose, for every $i \in \mathbb{N}$, a functional $x_i^* \in B_{X^*}$ such that

$$\left| \sum_{m\in F_i} a_m x_i^*(x_m) \right| > \delta_0,$$

and let

$$G_i = \{m \in F_i : |x_i^*(x_m)| \geq \frac{\delta_0}{2}\}.$$

It now follows, as G_i belongs to $\mathfrak{F}_{\frac{\delta_0}{2}}(s)[M]$, that there exists, for every $i \in \mathbb{N}$,

$$H_i \subset G_i,\ H_i \in S_\xi, \text{ such that } \left| \sum_{m\in H_i} a_m x_i^*(x_m) \right| > \frac{\delta_0}{2n_0} = \delta.$$

By extending H_i to a maximal S_ξ subset of $H_i \cup A_{\max F_i}$, if necessary, we can assume without loss of generality, that H_i itself is a maximal S_ξ subset of N, for all $i \in \mathbb{N}$. We also extend the sequence $(a_m)_{m\in N_1}$ to a scalar sequence $(a_m)_{m\in N}$ in the obvious manner.

Concluding, there exist $H_1 < H_2 < \cdots$ maximal S_ξ subsets of N and a scalar sequence $(a_m)_{m\in N}$ with $(a_m x_m)_{m\in N}$ series bounded by 1, so that for every $i \in \mathbb{N}$,

$$a_m \geq 0, \quad \text{for all } m \in H_i, \sum_{m\in H_i} a_m \leq 1 \quad \text{and} \quad \left\| \sum_{m\in H_i} a_m x_m \right\| > \delta.$$

Finally, choose $L \in [N]$ such that $F_i^{\xi}(L) = H_i$, for all $i \in \mathbb{N}$, and we are done. □

Definition 4.4. Let $M \in [\mathbb{N}]$, $L \in [M]$, $n \in \mathbb{N}$ and $\delta > 0$. The scalar sequence $(a_m)_{m\in M}$ is called ξ-faithful for (δ, L, n), provided the following conditions hold:

1. $|a_m| \leq \frac{\delta}{2}$, for all $m \in M$.
2. $(a_m x_m)_{m\in M}$ is series bounded by 1.

3. For every $i \leq n$ we have that

$$a_m \geq 0, \quad \text{for all } m \in F_i^\xi(L), \sum_{m \in F_i^\xi(L)} a_m \leq 1 \quad \text{and} \left\| \sum_{m \in F_i^\xi(L)} a_m x_m \right\| \geq \delta.$$

Lemma 4.5. *Let $s = (x_n)$ have no subsequence which is b.c.c. Let $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $\delta > 0$ satisfy the conclusion of Lemma 4.3 applied on s.*

1. *If $\xi = 0$, then some subsequence of $(x_m)_{m \in M}$ is equivalent to the unit vector basis of c_0.*
2. *If $\xi \geq 1$, then for every $N \in [M]$ there exists $P \in [N]$ so that for all $L \in [P]$ and $n \in \mathbb{N}$ there exists a scalar sequence $(a_m)_{m \in N}$ which is ξ-faithful for (δ, L, n).*

Proof. If $\xi = 0$, then according to Lemma 4.3 there exist a scalar sequence $(a_m)_{m \in M}$ and $L \in [M]$, $L = (l_i)$, so that $(a_m x_m)_{m \in M}$ is series bounded by 1 and $a_m > \delta$, for all $m \in L$. It now follows that if $x^* \in B_{X^*}$, then for all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} |x^*(x_{l_i})| \leq \frac{2C(\delta)}{\delta}$$

and thus (x_{l_i}) is equivalent to the unit vector basis of c_0.

Assume now that $\xi \geq 1$. Let $N \in [M]$ and $Q \in [N]$. It follows that no subsequence of $(x_m)_{m \in M}$ is equivalent to the unit vector basis of c_0 and therefore we obtain, by Corollary 3.3, $N_0 \in [Q]$ such that $(x_m)_{m \in N_0}$ is semi-boundedly complete.

We next choose, according to Lemma 4.3, a scalar sequence $(a_m)_{m \in N_0}$ and $L_0 \in [N_0]$ with $(a_m x_m)_{m \in N_0}$ series bounded by 1 so that for all $i \in \mathbb{N}$,

$$a_m \geq 0, \quad \text{for all } m \in F_i^\xi(L_0), \sum_{m \in F_i^\xi(L_0)} a_m \leq 1 \quad \text{and} \quad \left\| \sum_{m \in F_i^\xi(L_0)} a_m x_m \right\| \geq \delta.$$

Since $\lim_{m \in N_0} a_m = 0$, there exists $i_0 \in \mathbb{N}$ such that $|a_m| < \frac{\delta}{2}$, for all $m \in N_0$, $m > \max F_{i_0}^\xi(L_0)$. Set

$$N_1 = \{m \in N_0 : m > \max F_{i_0}^\xi(L_0)\} \quad \text{and} \quad L = \bigcup_{i > i_0} F_i^\xi(L_0).$$

We also extend the sequence $(a_m)_{m \in N_1}$ to a sequence $(a_m)_{m \in N}$ in the obvious manner. Evidently, $(a_m)_{m \in N}$ is ξ-faithful for (δ, L, n), for all $n \in \mathbb{N}$. We next consider the set

$$\mathcal{D} = \{L \in [N] : \forall n \in \mathbb{N} \quad \exists (a_m)_{m \in N} \quad \xi \text{ -faithful for } (\delta, L, n)\}$$

which is of course closed in $[N]$ and therefore Ramsey. Our previous argument yields the existence of $P \in [N]$ such that $[P] \subset \mathcal{D}$. Clearly, P satisfies the conclusion of part 2 of this lemma. □

Lemma 4.6. *Let $s = (x_n)$ be a weakly null sequence and assume that (x_n^*), the sequence of functionals biorthogonal to (x_n) is weakly null in $[(x_n)]^*$. Let $\xi < \omega_1$, $M \in [\mathbb{N}]$, $\epsilon > 0$ and $\delta > 0$. There exists $N \in [M]$ satisfying the following property: If $L \in [N]$, $n \in \mathbb{N}$ and there exists a scalar sequence $(a_m)_{m \in M}$ which is ξ-faithful*

for (δ, L, n), then there exists $(b_m)_{m\in M}$ which is ξ-faithful for (δ, L, n) and so that

$$\sum_{m\in N\setminus\bigcup_{i=1}^n F_i^\xi(L)} |b_m| < \epsilon.$$

Proof. The proof is similar to those of Lemmas 3.2 and 3.4. First choose a sequence $(\epsilon_i)_{i=0}^\infty$ of positive scalars such that $\sum_{i=0}^\infty \epsilon_i < \epsilon$. Using the same notation and terminology as in Lemma 3.2, let $n \in \mathbb{N}\cup\{0\}$ and $F \subset T_n$. The n-tuple of elements of M $(r_i)_{i\in T_n}$ and the infinite subset L of M, $L = (l_i)$, are said to satisfy property $(F - A_n)$ provided that $r_n < l_1$, if $n \geq 1$, and the following statement holds:

If $k \in \mathbb{N}$ and there exists a scalar sequence $(a_m)_{m\in M}$ which is ξ-faithful for $(\delta, \{r_i : i \in T_n \setminus F\} \cup \{l_i : i \geq 2\}, k)$, then there exists $(b_m)_{m\in M}$ which is ξ-faithful for $(\delta, \{r_i : i \in T_n \setminus F\} \cup \{l_i : i \geq 2\}, k)$ and such that

$$\sum_{i\in F} |b_{r_i}| + |b_{l_1}| < \sum_{i=0}^n \epsilon_i.$$

Let us also say that $(r_i)_{i\in T_n}$ and L satisfy property (A_n), if they satisfy property $(F - A_n)$, for all $F \subset T_n$.

We shall inductively construct an increasing sequence $(r_n)_{n=1}^\infty$ of elements of M and a decreasing sequence $(M_n)_{n=0}^\infty$ of infinite subsets of M with $r_n \in M_{n-1}$, if $n \geq 1$, so that if $n \in \mathbb{N}\cup\{0\}$ and $L \in [M_n]$, then $(r_i)_{i\in T_n}$ and L satisfy property (A_n).

The first inductive step is similar to the general one and so we shall not discuss it. Now we assume that $r_1 < \cdots < r_n$ and $M_0 \supset \cdots \supset M_n$ have been constructed with $r_i \in M_{i-1}$, if $i \geq 1$, so that if $i \leq n$, and $L \in [M_i]$, then $(r_j)_{j\in T_i}$ and L satisfy property (A_i). Let $r_{n+1} = \min M_n$ and fix $F \subset T_{n+1}$. We define

$$\Delta_F = \{L \in [M_n] : (r_i)_{i\in T_{n+1}} \text{ and } L \text{ satisfy } (F - A_{n+1})\}.$$

Clearly, Δ_F is closed in $[M_n]$ and therefore Ramsey. Suppose that for some $P \in [M_n]$, $P = (p_i)$, we had that $[P] \cap \Delta_F = \emptyset$. Let $q \in \mathbb{N}$ and set

$$L_j = \{p_j\} \cup \{p_i : i > q\}, \text{ for all } j \leq q.$$

Since $L_j \notin \Delta_F$, for all $j \leq q$, there exist integers $(k_j)_{j=1}^q$ as well as scalar sequences $(a_m^j)_{m\in M}$, $j \leq q$, so that letting $R = \{r_i : i \in T_{n+1} \setminus F\} \cup \{p_i : i > q\}$ we have that for all $j \leq q$,

$$(a_m^j)_{m\in M} \text{ is } \xi \text{ -faithful for } (\delta, R, k_j),$$

and moreover, if $(b_m)_{m\in M}$ is a ξ-faithful scalar sequence for (δ, R, k_j), then

$$\sum_{i\in F} |b_{r_i}| + |b_{p_j}| \geq \sum_{i=0}^{n+1} \epsilon_i.$$

Let $k_0 = \max\{k_j : j \leq q\}$ and observe that any scalar sequence $(b_m)_{m\in M}$ which is ξ-faithful for (δ, R, k_0), is also ξ-faithful for (δ, R, k_j), for all $j \leq q$. Next, let $t = \max F$ and note that

$$R = \{r_i : i \in T_{t-1} \setminus F\} \cup \{r_i : t < i \leq n+1\} \cup \{p_i : i > q\}.$$

By the induction hypothesis, since $(r_i)_{i=1}^{t-1}$ and $\{r_i : t \leq i \leq n+1\} \cup \{p_i : i > q\}$ satisfy property (A_{t-1}), there exists a scalar sequence $(b_m)_{m\in M}$ which is ξ-faithful

for (δ, R, k_0) and such that

$$\sum_{i\in F\setminus\{t\}} |b_{r_i}| + |b_{r_t}| < \sum_{i=0}^{t-1} \epsilon_i.$$

Thus,

$$\sum_{i\in F} |b_{r_i}| < \sum_{i=0}^{n} \epsilon_i.$$

It follows now by our previous observation, that $(b_m)_{m\in M}$ is ξ-faithful for (δ, R, k_j), for all $j \leq q$ and thus,

$$\sum_{i\in F} |b_{r_i}| + |b_{p_j}| \geq \sum_{i=0}^{n+1} \epsilon_i, \text{ for all } j \leq q.$$

Hence, $|b_{p_j}| \geq \epsilon_{n+1}$, for all $j \leq q$.

Now choose $m_0 > p_q$, $m_0 \in M$. Then, $|x^*_{p_j}(\sum_{m\leq m_0} b_m x_m)| \geq \epsilon_{n+1}$, for all $j \leq q$. But $\|\sum_{m\leq m_0} b_m x_m\| \leq 1$ and q is arbitrary and so $(x^*_{p_i})$ is not weakly null in $[(x_i)]^*$, contradicting our assumption.

Concluding, there exists $L \in [M_n]$ such that $[L] \subset \Delta_F$. By repeating the previous argument successively over all possible subsets of T_{n+1}, we obtain $M_{n+1} \in [M_n]$ such that $[M_{n+1}] \subset \Delta_F$, for all $F \subset T_{n+1}$. The inductive construction is now complete. Set $N = (r_i)$. Let $L \in [N]$ and $k \in \mathbb{N}$. Let $(a_m)_{m\in M}$ be a scalar sequence which is ξ-faithful for (δ, L, k). Suppose that $r_p = \max F^\xi_k(L)$ and let $G_q = \{i \in T_q : r_i \notin \bigcup_{j=1}^k F^\xi_j(L)\}$, for all $q \geq p$. Our construction yields that $(r_i)_{i=1}^q$ and $\{r_i : i > q\}$, satisfy $(G_q - A_q)$, for all $q \geq p$. We also have, by stability, that for all $q \geq p$,

$$F^\xi_j(L) = F^\xi_j(R_q), \text{ for all } j \leq k$$

where, $R_q = \{r_i : i \in T_q \setminus G_q\} \cup \{r_i : i > q+1\}$, and therefore there exists a scalar sequence $(b^q_m)_{m\in M}$ which is ξ-faithful for (δ, L, k) and such that

$$\sum_{i\in G_q} |b^q_{r_i}| < \sum_{i=0}^{q} \epsilon_i < \epsilon.$$

Thus,

$$\sum_{m\in N\setminus\bigcup_{i=1}^k F^\xi_i(L),\, m\leq r_q} |b^q_m| < \epsilon.$$

Finally, let $(b_m)_{m\in M}$ be any cluster point of the sequence $\big((b^q_m)_{m\in M}\big)_{q\geq p}$ in $[-1,1]^M$. Evidently, this is the desired sequence. □

Lemma 4.7. *Let $t = (y_i)$ be a sequence in B_X, $\xi < \omega_1$ and μ a finitely supported probability measure on $\mathbb{N}$. Assume there exist $x^* \in B_{X^*}$ and $\epsilon > 0$ so that $x^*(\mu\cdot t) \geq 2\epsilon$. Assume further that there exists $l \in \mathbb{N}$, $l < \operatorname{supp}\mu$, such that $l\|\mu\|_{\zeta_i} < \epsilon$, for every $i \leq l$, where $(\zeta_i + 1)$ is the sequence of ordinals associated to ξ. Let $E = \{n \in \operatorname{supp}\mu : x^*(y_n) \geq \frac{1}{2}x^*(\mu\cdot t)\}$. Then, there exists a maximal S_ξ set containing l and contained in $\{l\} \cup E$.*

Proof. Assume on the contrary, that no subset of $\{l\} \cup E$ containing l is a maximal S_ξ set. We then claim that $\{l\} \cup E$ belongs to S_ξ. Indeed, suppose that $\{l\} \cup E = \{m_1, \cdots, m_k\}$, $m_1 = l$, and choose $r \leq k$ maximal with respect to $\{m_1, \cdots, m_k\} \in S_\xi$. If $r < k$, then Lemma 2.5 yields that $\{m_1, \cdots, m_r\}$ is a maximal S_ξ set contradicting our assumption. Thus $r = k$ as claimed.

There exists now $i \leq l$ such that $\{l\} \cup E$ belongs to S_{ζ_i+1}. It follows that we can find $p \leq l$ and consecutive S_{ζ_i} sets $(A_j)_{j=1}^p$ so that

$$\{l\} \cup E = \bigcup_{j=1}^{p} A_j.$$

But now,

$$\begin{aligned} x^*(\mu \cdot t) &= \sum_{n \in E} \mu(n) x^*(y_n) + \sum_{n \in \operatorname{supp} \mu \setminus E} \mu(n) x^*(y_n) \\ &\leq \sum_{j=1}^{p} \mu(A_j) + \frac{1}{2} x^*(\mu \cdot t) \\ &\leq p \|\mu\|_{\zeta_i} + \frac{1}{2} x^*(\mu \cdot t) \end{aligned}$$

and thus $x^*(\mu \cdot t) < 2\epsilon$. This contradiction completes the proof of the lemma. □

We are now ready for the proof of the main result of this paper.

Proof of Theorem 1.5. Assume that s has no subsequence which is b.c.c. Choose $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $\delta > 0$ satisfying the conclusion of Lemma 4.5 applied on s. If $\xi = 0$, we are done since some subsequence of s is equivalent to the unit vector basis of c_0.

Assume now that $\xi \geq 1$. Choose according to Corollary 3.3, $M_1 \in [M]$ so that the sequence $(x_m^*)_{m \in M_1}$ of functionals biorthogonal to $(x_m)_{m \in M_1}$ is weakly null in $[(x_m)_{m \in M_1}]^*$. Next, choose according to Lemma 2.4, $M_2 \in [M_1]$ such that

$$F \setminus \{\min F\} \in S_\xi(M_1), \text{ for all } F \in S_\xi[M_2]. \tag{1}$$

Let $0 < \epsilon < 1$ and choose a sequence of positive scalars (ϵ_n) such that $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. Choose also $\lambda \geq \frac{4}{\delta}(1 + 4\epsilon)$. Lemma 3.2 now yields $M_3 \in [M_2]$ such that for every $L \in [M_3]$ and $n \in \mathbb{N}$, if there exists $x^* \in B_{X^*}$ which is ξ-good for (δ, L, n), then there exists $y^* \in B_{X^*}$, ξ-good for (δ, L, n) and such that

$$\sum_{m \in M_3 \setminus \bigcup_{i=1}^{n} \operatorname{supp} \xi_i^L} |y^*(x_m)| < \epsilon. \tag{2}$$

We continue our choice of infinite subsets of M by choosing $M_4 \in [M_3]$ according to Lemma 4.6. Thus, for every $L \in [M_4]$ and $n \in \mathbb{N}$, if there exists a scalar sequence $(a_m)_{m \in M_3}$ which is ξ-faithful for (δ, L, n), then there exists $(b_m)_{m \in M_3}$, ξ-faithful for (δ, L, n), and such that

$$\sum_{m \in M_4 \setminus \bigcup_{i=1}^{n} F_i^\xi(L)} |b_m| < \epsilon. \tag{3}$$

Finally, choose $M_5 \in [M_4]$ according to Lemma 4.5 applied for "N" $= M_4$. It follows now, by (3), that for every $L \in [M_5]$ and $n \in \mathbb{N}$, there exists a scalar

sequence $(a_m)_{m\in M_3}$ which is ξ-faithful for (δ, L, n) and such that

$$\sum_{m\in M_4\setminus\bigcup_{i=1}^n F_i^\xi(L)} |a_m| < \epsilon. \tag{4}$$

Now let $Q \in [M_5]$. Let $(\zeta_n + 1)$ be the sequence of ordinals associated to ξ. Repeated applications of Proposition 2.15 now yield an increasing sequence of elements of Q, (l_i), and a sequence (Q_i) of infinite subsets of Q, so that

$$l_1 < \operatorname{supp} \xi_1^{Q_1} < l_2 < \operatorname{supp} \xi_1^{Q_2} < \cdots$$

and

$$l_i \|\xi_1^{Q_i}\|_\alpha < \epsilon_i, \text{ for all } \alpha \in \{\zeta_m : m \le l_i\} \cup \{0\} \text{ and } i \in \mathbb{N}.$$

We thus obtain, by stability, $P \in [Q]$ such that $\xi_i^P = \xi_1^{Q_i}$, for all $i \in \mathbb{N}$, and therefore,

$$\begin{aligned} &l_1 < \operatorname{supp} \xi_1^P < l_2 < \operatorname{supp} \xi_2^P < \cdots \\ &l_i \|\xi_i^P\|_\alpha < \epsilon_i, \text{ for all } \alpha \in \{\zeta_m : m \le l_i\} \cup \{0\} \text{ and } i \in \mathbb{N}. \end{aligned} \tag{5}$$

Let K be the ℓ_1^ξ spreading model constant of $(x_m)_{m\in M}$. By our choice of $M_2 \in [M_1]$, (1) and (5) yield that

$$\|\xi_i^P \cdot s\| \ge K - \frac{\epsilon_i}{l_i}(K+1)$$

and hence $(\xi_i^P \cdot s)$ is semi-normalized.

We now claim that

$$\left\| \sum_{i\in F} \xi_i^P \cdot s \right\| \le 2\lambda,$$

for all $F \in [\mathbb{N}]^{<\infty}$. Our claim of course implies that $(\xi_i^P \cdot s)$ is equivalent to the unit vector basis of c_0. Were our claim false, there would exist $q \in \mathbb{N}$, integers $i_1 < \cdots < i_q$, and $y^* \in B_{X^*}$ such that

$$y^*(\xi_{i_n}^P \cdot s) \ge 0, \text{ for } n \le q, \text{ and } \sum_{n=1}^q y^*(\xi_{i_n}^P \cdot s) > \lambda.$$

Since $P \in [M_3]$, (2) yields $x^* \in B_{X^*}$ such that

$$\begin{aligned} &x^*(\xi_{i_n}^P \cdot s) \ge 0, \text{ for } n \le q, \sum_{n=1}^q x^*(\xi_{i_n}^P \cdot s) > \lambda \text{ and} \\ &\sum_{m\in M_3\setminus\bigcup_{n=1}^q \operatorname{supp} \xi_{i_n}^P} |x^*(x_m)| < \epsilon. \end{aligned} \tag{6}$$

We set $I = \{n \le q : x^*(\xi_{i_n}^P \cdot s) \ge 2\epsilon_{i_n}\}$. It follows that

$$\sum_{n\in I} x^*(\xi_{i_n}^P \cdot s) > \lambda - 2\epsilon. \tag{7}$$

We now fix $n \in I$ and let

$$E_n = \left\{ m \in \operatorname{supp} \xi_{i_n}^P : x^*(x_m) \ge \frac{1}{2} x^*(\xi_{i_n}^P \cdot s) \right\}.$$

Because of (5), Lemma 4.7 yields the existence of a maximal S_ξ subset D_n of $\{l_{i_n}\} \cup E_n$ such that $\min D_n = l_{i_n}$. Let $m_0 = \max D_{\max I}$ and set

$$L = \bigcup_{n \in I} D_n \cup \{m \in Q : m > m_0\}.$$

Since $L \in [M_5]$, we obtain through (4), a scalar sequence $(a_m)_{m \in M_3}$ which is ξ-faithful for $(\delta, L, |I|)$ and such that

$$\sum_{m \in M_4 \setminus \bigcup_{n \in I} D_n} |a_m| < \epsilon. \tag{8}$$

We recall here that for every $n \in I$,

$$a_m \geq 0, \quad \text{for all } m \in D_n, \sum_{m \in D_n} a_m \leq 1 \quad \text{and} \quad \left\| \sum_{m \in D_n} a_m x_m \right\| > \delta.$$

Of course, $(a_m x_m)_{m \in M_3}$ is series bounded by 1 and $|a_m| \leq \frac{\delta}{2}$, for all $m \in M_3$. Therefore,

$$\sum_{m \in D_n \setminus \{l_{i_n}\}} a_m > \frac{\delta}{2}, \text{ for all } n \in I. \tag{9}$$

Our construction yields that

$$\sum_{m \in M_3 \setminus \bigcup_{n \in I} D_n} |a_m||x^*(x_m)| < 2\epsilon. \tag{10}$$

Indeed,

$$\begin{aligned} \sum_{m \in M_3 \setminus \bigcup_{n \in I} D_n} |a_m||x^*(x_m)| &\leq \sum_{m \in M_4 \setminus \bigcup_{n \in I} D_n} |a_m| + \sum_{m \in M_3 \setminus M_4} |x^*(x_m)| \\ &< \epsilon + \sum_{m \in M_3 \setminus \bigcup_{n=1}^{q} \operatorname{supp} \xi_{i_n}^P} |x^*(x_m)|, \quad \text{by (8)}, \\ &\text{and since } P \in [M_4], < \epsilon + \epsilon = 2\epsilon, \quad \text{by (6)}. \end{aligned}$$

We also observe that

$$\sum_{n \in I} |x^*(x_{l_{i_n}})| < \epsilon, \text{ as } l_{i_n} \in M_3 \setminus \bigcup_{n=1}^{q} \operatorname{supp} \xi_{i_n}^P, \text{ for all } n \in I. \tag{11}$$

Hence,

$$\begin{aligned}
x^*\Big(\sum_{m\le m_0} a_m x_m\Big) &\ge \sum_{m\in\bigcup_{n\in I} D_n} a_m x^*(x_m) - \sum_{m\in M_3\setminus\bigcup_{n\in I} D_n} |a_m||x^*(x_m)| \\
&> \sum_{n\in I}\sum_{m\in D_n} a_m x^*(x_m) - 2\epsilon, \text{ by (10)}, \\
&\ge \sum_{n\in I}\sum_{m\in D_n\setminus\{l_{i_n}\}} a_m x^*(x_m) - \sum_{n\in I} |x^*(x_{l_{i_n}})| - 2\epsilon \\
&\ge \sum_{n\in I}\sum_{m\in D_n\setminus\{l_{i_n}\}} a_m \frac{1}{2} x^*(\xi^P_{i_n}\cdot s) - 3\epsilon, \text{ by (11)}, \\
&\text{and since } D_n\setminus\{l_{i_n}\}\subset E_n, \\
&\ge \frac{\delta}{4}\sum_{n\in I} x^*(\xi^P_{i_n}\cdot s) - 3\epsilon, \text{ by (9)} \\
&\ge \frac{\delta}{4}(\lambda - 2\epsilon) - 3\epsilon, \text{ by (7)}.
\end{aligned}$$

It now follows that

$$1 \ge \Big\|\sum_{m\le m_0} a_m x_m\Big\| > \frac{\delta}{4}(\lambda - 2\epsilon) - 3\epsilon,$$

and thus $\lambda < \frac{4}{\delta}(4\epsilon + 1)$ contradicting the choice of λ.

Hence, our claim holds and $(\xi^P_i \cdot s)$ is equivalent to the unit vector basis of c_0. Moreover, the equivalence constant C, depends only on K, δ and ϵ. It is now easily seen that the set

$$\{P \in [M_5] : (\xi^P_i \cdot s) \text{ is } C\text{-equivalent to the } c_0\text{-basis }\},$$

is closed in $[M_5]$ and therefore Ramsey. Our previous argument yields $N \in [M_5]$ so that $(\xi^Q_i \cdot s)$ is equivalent to the unit vector basis of c_0, for all $Q \in [N]$. The proof of Theorem 1.5 is now complete. □

Theorem 1.4 follows immediately from Theorem 1.5.

5. Non-trivial weak Cauchy sequences

The last section is devoted to the relation between Theorems 1.4 and 1.6. We first observe the following immediate consequence of Theorem (1.6)

Corollary 5.1. *The following are equivalent for a non-trivial weak Cauchy sequence (x_n) in a Banach space:*

1. *There exists a subsequence of (x_n) which is (s.s.).*
2. *There exists a subsequence (x_{m_n}) of (x_n) such that every convex block subsequence of (x_{m_n}) is semi-boundedly complete.*

Proof. The fact that 1 implies 2 is immediate since (s.s.) sequences are easily seen to be semi-boundedly complete, and every convex block subsequence of an (s.s) sequence is also (s.s) [R].

Suppose now that 2 holds. If no subsequence of (x_n) is (s.s), then Theorem (1.6) yields a convex block subsequence of (x_{m_n}) equivalent to the summing basis. But the summing basis is not semi-boundedly complete. This contradiction shows that 1 must hold. □

Corollary 5.1 yields the following equivalent formulation of Rosenthal's theorem:

Corollary 5.2. *For a non-trivial weak Cauchy sequence (x_n) in a Banach space, one of the following statements holds exclusively:*

1. *There exists a subsequence (x_{m_n}) of (x_n) such that every convex block subsequence of (x_{m_n}) is semi-boundedly complete.*
2. *Every subsequence of (x_n) admits a convex block subsequence equivalent to the summing basis.*

Evidently, Corollary 5.2 makes even more transparent the analogy between Rosenthal's result and Theorem 1.4. We also note here that, as it is shown in [R], (x_n) is an (s.s) sequence, if and only if every proper subsequence of its difference sequence is semi-boundedly complete. (In the terminology of [R], the difference sequence (e_i) of an (s.s) sequence is (c.c.). That is, if $\sup_n \|\sum_{i=1}^n a_i e_i\|$ is finite, then the scalar sequence (a_n) converges.)

We next give a quantitative version of Theorem 1.6.

Corollary 5.3. *Let $t = (x_n)$ be a non-trivial weak Cauchy sequence having no subsequence which is (s.s.). Then for every $N \in [\mathbb{N}]$ there exist $M \in [N]$, a countable ordinal ξ and a constant $C > 0$, so that $(\xi_n^L \cdot t)$ is C-equivalent to the summing basis for every $L \in [M]$.*

The proof of this corollary requires the following lemma.

Lemma 5.4. *Let ξ be a countable ordinal and $P \in [\mathbb{N}]$. Let $t = (x_n)$ be a sequence in a Banach space which is (L, ξ) convergent (Definition 2.2), for every $L \in [P]$. Given (ϵ_n), a sequence of positive scalars, there exists $M \in [P]$ such that $\|\xi_n^L \cdot t\| < \epsilon_n$, for every $L \in [M]$ and all $n \in \mathbb{N}$.*

Proof. Let $Q \in [P]$. Our assumptions allow us to choose (k_n), an infinite subset of $\mathbb{N}$, so that $\|\xi_{k_n}^Q \cdot t\| < \epsilon_n$, for all $n \in \mathbb{N}$. Stability now yields $L \in [Q]$ such that $\|\xi_n^L \cdot t\| < \epsilon_n$, for all $n \in \mathbb{N}$. The assertion of the lemma follows from this since $\{L \in [P] : \|\xi_n^L \cdot t\| < \epsilon_n, \text{ for all } n \in \mathbb{N}\}$ is a closed subset of $[P]$ and therefore Ramsey. □

Proof of Corollary 5.3. Theorem (1.6) yields $u = (u_n)$ a convex block subsequence of (x_n) equivalent to the summing basis. We set $v = (x_n - u_n)$ which is clearly a weakly null sequence. Employing the results of [A-A], we find a countable ordinal ξ such that no subsequence of v is an ℓ_1^ξ spreading model. It follows now, by Theorem 1.7, that there exists $P \in [N]$ such that $\lim_n \|\xi_n^L \cdot v\| = 0$, for every $L \in [P]$. We next choose (ϵ_n), a sequence of positive scalars such that $\sum_n \epsilon_n < 1$. As a consequence of Lemma 5.4 we obtain $M \in [P]$ such that $\|\xi_n^L \cdot v\| < \epsilon_n$, for every $L \in [M]$ and all $n \in \mathbb{N}$. A standard perturbation result now yields $D > 0$ such that $(\xi_n^L \cdot t)$ is D-equivalent to $(\xi_n^L \cdot u)$, for all $L \in [M]$. Since the summing basis is uniformly equivalent to all of its convex block subsequences, we obtain $C > 0$ such that $(\xi_n^L \cdot t)$ is C-equivalent to the summing basis, for all $L \in [M]$. This completes the proof. □

We observe the similarity between the statements of Theorem 1.5 and Corollary 5.3. However, the set of ordinals ξ satisfying the conclusion of Theorem 1.5 is a bounded segment of $[0, \omega_1)$, in contrast with the corresponding set in Corollary 5.3 which is of course unbounded.

We continue our discussion about the relation between Theorems 1.4 and 1.6. Recall that Rosenthal's c_0-theorem states that every non-trivial weak Cauchy sequence (x_j), either has an (s.s) subsequence, or else there exists a convex block subsequence (s_j) of (x_j) equivalent to the summing basis. In the later case, setting $v_j = x_j - s_j$, we observe that (v_j) is weakly null and that $(x_j - v_j)$ is equivalent to the summing basis. Therefore, passing to a convex block subsequence in Rosenthal's theorem acts as a filtration to remove the "noise" coming from an arbitrary weakly null sequence. In our case the reasoning for passing to a convex block subsequence is different: It exhausts the local ℓ_1 structure of the sequence.

In spite of these differences it seems that there are similarities in the statements for weakly null and non-trivial weak Cauchy sequences. Our final corollary which is the analog to Elton's dichotomy illustrates this.

Corollary 5.5. *For a non-trivial weak Cauchy sequence (x_n) one of the following statements holds exclusively:*

1. *Every subsequence of (x_n) admits a subsequence equivalent to the summing basis.*
2. *There exists a subsequence of (x_n) which is semi-boundedly complete.*

Proof. Clearly the statements are mutually exclusive since the summing basis is not semi-boundedly complete.

Suppose that 2. does not hold. It follows that no subsequence of (x_n) is (s.s). Theorem 1.6 now yields that every subsequence of (x_n) admits a convex block subsequence equivalent to the summing basis. To prove that 1 holds let (x_{m_n}) be a subsequence of (x_n). Without loss of generality, by passing to a subsequence according to Proposition 2.2 of [R], we can assume that (x_{m_n}) dominates the summing basis.

Next choose (u_n), a convex block subsequence of (x_n) equivalent to the summing basis. We set $y_n = x_{m_n} - u_n$, for all $n \in \mathbb{N}$. If (y_n) is not semi-normalized, then 1 follows. So assuming that (y_n) is semi-normalized we claim that there exists a subsequence of (y_n) equivalent to the unit vector basis of c_0. Indeed, if that were not the case, then by Elton's dichotomy, Corollary 3.3, there would exist a semi-boundedly complete subsequence of (y_n). But since every subsequence of (x_{m_n}) dominates the summing basis (and therefore every subsequence of (u_n) as well), we obtain that (x_{m_n}) has a semi-boundedly complete subsequence which of course contradicts our assumption that 2 does not hold. Hence, our claim must hold and it immediately yields a subsequence of (x_{m_n}) equivalent to the summing basis in view of the following elementary fact: Let (f_n) and (g_n) be sequences in a Banach space with (f_n) equivalent to the summing basis and (g_n) equivalent to the c_0 basis. Then there exists a subsequence of $(f_n + g_n)$ equivalent to the summing basis. $\square$

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Department of Mathematics, University of Athens, Athens 15784, Greece

Current address: Department of Mathematics, National Technical University of Athens, 15780 Athens, Greece

E-mail address: sargyros@math.ntua.gr

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078-1058

E-mail address: ioagaspa@math.okstate.edu